$\mathcal{O}$-OPERATORS OF LODAY ALGEBRAS AND ANALOGUES OF THE CLASSICAL YANG-BAXTER EQUATION

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ABSTRACT. We introduce notions of $\mathcal{O}$-operators of the Loday algebras including the dendriform algebras and quadri-algebras as a natural generalization of Rota-Baxter operators. The invertible $\mathcal{O}$-operators give a sufficient and necessary condition on the existence of the 2^{n+1} operations on an algebra with the 2^n operations in an associative cluster. The analogues of the classical Yang-Baxter equation in these algebras can be understood as the $\mathcal{O}$-operators associated to certain dual bimodules. As a byproduct, the constraint conditions (invariances) of nondegenerate bilinear forms on these algebras are given.

CONTENTS

1. INTRODUCTION

Dendriform algebras are equipped with an associative product which can be written as a linear combination of nonassociative compositions. They were introduced by Loday ([Lo1]) in 1995 with motivation from algebraic K-theory and have been studied quite extensively with connections to several areas in mathematics and physics, including operads ([Lo3]), homology

2000 *Mathematics Subject Classification.* 16W30, 17A30, 17B60.
Key words and phrases. Associative algebra, dendriform algebra, quadri-algebra, Yang-Baxter equation.

([Fr1-2]), Hopf algebras ([Ch], [H1-2], [Ron], [LR2]), Lie and Leibniz algebras ([Fr2]), combinatorics ([LR1]), arithmetic([Lo2]) and quantum field theory ([Fo]) and so on (see [EMP] and the references therein).

Later quite a few more similar algebra structures have been introduced, such as quadri-algebras of Aguiar and Loday ([AL]) and octo-algebras of Leroux ([Le3]). All of them are called Loday algebras ([V], or ABQR operad algebras in [EG1-2]). These algebras have a common property of "splitting associativity", that is, expressing the multiplication of an associative algebra as the sum of a string of binary operations ([Lo3]). It is also called an (associative) cluster of operations by Leroux ([Le1-3]). Explicitly, let $(X, *)$ be an associative algebra over a field $\mathbb{F}$ of characteristic zero and $(*_i)_{1\leq i\leq N} : X\otimes X \to X$ be a family of binary operations on X. Then the operation $*$ splits into the N operations $*_1, \cdots, *_N$ or the operation $*$ is a cluster of N (binary) operations if

$$x * y = \sum_{i=1}^{N} x *_i y, \quad \forall x, y \in X. \tag{1.1}$$

In this paper, we will pay our main attention to the case that the number N of the operations in the string is 2^n (in fact, there are the dendriform trialgebras with $N = 3$ ([E1], [LR3]), the ennea-algebras with $N = 9$ ([Le1]) and so on).

Of course, besides the relation (1.1), $*_i$ should satisfy certain additional conditions. In fact, there are several quite different motivations to introduce the Loday algebras (including dendriform algebras, quadri-algebras and octo-algebras) whose operation axioms can be summarized to be a set of "associativity" relations ([EG1]). In this paper, we give an approach which emphasizes the bimodule (representation) structures of these algebras. For these known algebras, one can find that in an associative cluster the 2^{n+1} operations give a natural bimodule structure of an algebra with the 2^n operations on the underlying vector space of the algebra itself, which is the beauty of such algebra structures. Equivalently, the "rule" of constructing such algebras in an associative cluster is that, by induction, for the algebra $(A, *_i)_{1\leq i\leq 2^n}$, besides the natural (regular) bimodule of A on the underlying vector space of A itself given by the left and right multiplication operators, one can introduce the 2^{n+1} operations $\{*_{i_1}, *_{i_2}\}_{1\leq i\leq 2^n}$ such that

$$x *_i y = x *_{i_1} y + x *_{i_2} y, \quad \forall x, y \in A, \ 1 \leq i \leq 2^n, \tag{1.2}$$

and their left and right multiplication operators can give a bimodule of $(A, *_i)_{1\leq i\leq 2^n}$ by acting on the underlying vector space of A itself. For example, it is known that a dendriform algebra gives a natural bimodule of its associated associative algebra ([Lo1]).

These algebras have been studied extensively from many sides including certain operadic interpretation ([Le3], [EG1]). Obviously, it is quite important to consider how to construct a new type of algebras from the known algebras. As has been pointed out in [EG1], one of the main themes is the use of a linear operator with certain features. A successful example is the Rota-Baxter operators on a known type of algebras to obtain another type of algebras with richer structures. Rota-Baxter operators were introduced by G. Baxter ([Bax]) in 1960 and were

realized their importance by G.-C. Rota in combinatorics and other fields in mathematics ([Rot1-4]) and have been related to several areas of mathematics and physics ([At], [Mi1-2], [Ca], [Der], [Deb], [Ng], [CK], [E1-2], [EG1-2]). In fact, a Rota-Baxter operator on an associative algebra can be used to construct a dendriform algebra ([Ag2]), a Rota-Baxter operator on a dendriform algebra or a pair of commutating Rota-Baxter operators on an associative algebra can be used to construct a quadri-algebra ([AL]), a Rota-Baxter operator on a quadri-algebra or a set of three pairwise commutative Rota-Baxter operators on an associative algebras can be used to construct an octo-algebra ([Le3]).

In this paper, we give some more direct relationships between these Loday algebras. The key of our study is that we introduce a kind of operators, namely, $\mathcal{O}$-operators of those algebras which generalize the Rota-Baxter operators to all bimodules. In this sense, Rota-Baxter operators are just the $\mathcal{O}$-operators associated to the regular bimodules. The notion of $\mathcal{O}$-operator was introduced by Kupershmidt ([Ku]) for a Lie algebra and can be traced back to Bordemann ([Bo1]) in the study of integrable systems. It was also extended to be defined for an associative algebra ([BGN] and [Bai3]; such a structure also appeared independently in [U] under the name of generalized Rota-Baxter operator). A direct consequence in the case of the invertible $\mathcal{O}$-operators is to give a sufficient and necessary condition on the existence of the 2^{n+1} operations on an algebra with the 2^n operations in an associative cluster, which to our knowledge, it has not been written down explicitly yet.

On the other hand, one of the reasons that Kupershmidt introduced the $\mathcal{O}$-operators of a Lie algebra is to generalize (the operator form of) the famous classical Yang-Baxter equation in the Lie algebra ([Se], [Ku]). Even an $\mathcal{O}$-operator of a Lie algebra can give a solution of the classical Yang-Baxter equation in a larger Lie algebra ([Bai1]). So it is natural to consider its analogues in associative algebras and the Loday algebras. The associative Yang-Baxter equation was introduced by Aguiar ([Ag1]) to study the infinitesimal bialgebras given by Joni and Rota ([JR]) in order to provide an algebraic framework for the calculus of divided difference. Its relation with $\mathcal{O}$-operators of an associative algebra has been given in [BGN]. An analogue of the classical Yang-Baxter equation in a dendriform algebra was introduced in [Bai3] which was closely related to a kind of bialgebra structures on dendriform algebras (dendriform D-bialgebras). Both of these analogues can be interpreted in terms of $\mathcal{O}$-operators. With an adjustable symmetry, a solution of the classical Yang-Baxter equation in a Lie algebra and its analogues in an associative algebra or a dendriform algebra is just an $\mathcal{O}$-operator associated to certain "dual" bimodules of the regular bimodules. In this sense, the classical Yang-Baxter equation and their analogues are "dual" to the corresponding Rota-Baxter operators and it is also reasonable to extend such an idea to the other Loday algebras. We believe that all these analogues would play important roles in many fields as the classical Yang-Baxter equation in a Lie algebra has done ([Dr], [CP]).

Furthermore, the above study has a remarkable byproduct. It is known that the nondegenerate bilinear forms satisfying certain ("invariant") conditions are always important ([Bo2]), like the (symmetric) trace form and the (skew-symmetric) Connes cocycle ([Co]) on an associative algebra. However, sometimes, the "invariant" conditions are not easily known for certain algebras. Motivated by the Drinfeld's famous observation on the relation between the invertible (skew-symmetric) solutions of the classical Yang-Baxter equation and the 2-cocycles in a Lie algebra ([Dr]), the invertible solutions of the above analogues of the classical Yang-Baxter equation (with an adjustable symmetry) also can give some nondegenerate bilinear form satisfying certain conditions. By using the $\mathcal{O}$-operators again, one can find that these nondegenerate bilinear forms on an algebra with 2^n operations can give the splitting 2^{n+1} operations in an associative cluster. We will illustrate that, starting from the symmetric invariant (trace) forms on associative algebras, there is a chain of bilinear forms which corresponds to the cluster of operations.

The $\mathcal{O}$-operators of an associative algebra have been studied in [BGN] and [NBG] where the relations between the associative algebra and the compatible dendriform algebra structures were given. In this paper, we will give a detailed study on the other structures. We would like to point out that although many structures are similar and many results seem "reasonable", an explicit and rigorous proof is still necessary. It is also necessary to write down the details explicitly for a further development. Furthermore, it is likely that there is an operadic interpretation of the study in this paper. The paper is organized as follows. In Section 2, for self-containment, we recall the relationships between the $\mathcal{O}$-operators of associative algebras and dendriform algebras and associative Yang-Baxter equation. In Section 3, we study the $\mathcal{O}$-operators of dendriform algebras and their relationships with D-equation which is an analogue of the classical Yang-Baxter equation for a dendriform algebra and quadri-algebras. In section 4, we give a similar study on the $\mathcal{O}$-operator of quadri-algebras. In Section 5, we summarize the study in the previous sections and then give a general illustration of a similar study for any algebra with 2^n operations in an associative cluster.

Throughout this paper, all algebras are finite-dimensional and over a field of characteristic zero. We also give some notations as follows. Let A be an algebra with an operation $*$.

(1) Let $L_*(x)$ and $R_*(x)$ denote the left and right multiplication operator respectively, that is, $L_*(x)y = R_*(y)x = x * y$ for any $x, y \in A$. We also simply denote them by $L(x)$ and $R(x)$ respectively without confusion. Moreover let $L_*, R_* : A \to gl(A)$ be two linear maps with $x \to L_*(x)$ and $x \to R_*(x)$ respectively.

(2) Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. Set

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, r_{13} = \sum_i a_i \otimes 1 \otimes b_i,\ r_{23} = \sum_i 1 \otimes a_i \otimes b_i, \tag{1.3}$$

where 1 is a scale. The operation between two rs is in an obvious way. For example,

$$r_{12} * r_{13} = \sum_{i,j} a_i * a_j \otimes b_i \otimes b_j,\ r_{13} * r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i * b_j,\ r_{23} * r_{12} = \sum_{i,j} a_i \otimes a_j * b_i \otimes b_j, \tag{1.4}$$

and so on.

(3) For any linear map $\rho : A \to gl(V)$, define a linear map $\rho^* : A \to gl(V^*)$ by

$$\langle \rho(x)u, v^* \rangle = \langle u, \rho^*(x)v^* \rangle, \ \ \forall x \in A, \ u \in V, \ v^* \in V^*, \tag{1.5}$$

where $\langle , \rangle$ is the ordinary pair between the vector space A and its dual space A^*.

2. Preliminaries: $\mathcal{O}$-operators of associative algebras

For self-containment, we recall some facts on $\mathcal{O}$-operators of associative algebras. Most of the results can be found in [Ag1-4], [Bai3], [BGN], [NBG].

2.1. $\mathcal{O}$-operators of associative algebras and dendriform algebras.

Definition 2.1.1. Let A be an associative algebra and V be a vector space. Let $l, r : A \to gl(V)$ be two linear maps. V (or the pair (l, r), or (l, r, V)) is called a *bimodule* of A if

$$l(xy)v = l(x)l(y)v, \ r(xy)v = r(y)r(x)v, \ l(x)r(y)v = r(y)l(x)v, \ \forall\, x, y \in A, v \in V. \tag{2.1.1}$$

In fact, according to [Sc], (l, r, V) is a bimodule of an associative algebra A if and only if the direct sum $A \oplus V$ of vector spaces is turned into an associative algebra (the *semidirect sum*) by defining multiplication in $A \oplus V$ by

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1), \ \ \forall x_1, x_2 \in A, v_1, v_2 \in V. \tag{2.1.2}$$

We denote it by $A \ltimes_{l,r} V$.

Example 2.1.2. Let A be an associative algebra. If (l, r, V) is a bimodule of A, then (r^*, l^*, V^*) is a bimodule of A, which is called the dual bimodule of (l, r, V). In particular, both (L, R, A) and (R^*, L^*, A^*) are bimodules of A. The former is called a regular bimodule of A.

Definition 2.1.3. Let $(A, *)$ be an associative algebra and (l, r, V) be a bimodule. A linear map $T : V \to A$ is called an *$\mathcal{O}$-operator associated to* (l, r, V) if T satisfies

$$T(u) * T(v) = T(l(T(u))v + r(T(v))u), \ \ \forall u, v \in V. \tag{2.1.3}$$

Example 2.1.4. Let $(A, *)$ be an associative algebra. A linear map $R : A \to A$ is called a Rota-Baxter operator on A (of weight zero) if R is an $\mathcal{O}$-operator associated to the regular bimodule (L_*, R_*), that is, R satisfies ([Bax], [Rot1-4], etc.)

$$R(x) * R(y) = R(R(x) * y + x * R(y)), \ \ \forall\, x, y \in A. \tag{2.1.4}$$

Definition 2.1.5. Let A be a vector space with two bilinear products denoted by $\prec$ and $\succ$. $(A, \prec, \succ)$ is called a *dendriform algebra* if for any $x, y, z \in A$,

$$(x \prec y) \prec z = x \prec (y * z), \ \ (x \succ y) \prec z = x \succ (y \prec z), \ \ x \succ (y \succ z) = (x * y) \succ z, \tag{2.1.5}$$

where $x * y = x \prec y + x \succ y$.

Proposition 2.1.6. ([Lo1]) *Let $(A, \prec, \succ)$ be a dendriform algebra.*

(1) The product given by

$$x * y = x \prec y + x \succ y, \ \ \forall x, y \in A, \tag{2.1.6}$$

defines an associative algebra. We call $(A, *)$ the associated associative algebra of $(A, \succ, \prec)$ and $(A, \succ, \prec)$ is called a compatible dendriform algebra structure on the associative algebra $(A, *)$.

(2) $(L_{\succ}, R_{\prec})$ is a bimodule of the associated associative algebra $(A, *)$.

The following conclusion is obvious.

Corollary 2.1.7. *Let $(A, *)$ be an associative algebra and $\succ, \prec$ be two bilinear products on A. Then $(A, \succ, \prec)$ is a dendriform algebra if and only if equation (2.1.6) holds and $(L_{\succ}, R_{\prec})$ is a bimodule of $(A, *)$.*

Theorem 2.1.8. ([NBG]) *Let $T : V \to A$ be an $\mathcal{O}$-operator of an associative algebra $(A, *)$ associated to a bimodule (l, r, V). Then there is a dendriform algebra structure on V given by*

$$u \succ v = l(T(u))v, \ \ u \prec v = r(T(v))u, \ \ \forall\, u, v \in V. \tag{2.1.7}$$

Therefore, there exists an associated associative algebra structure on V given by equation (2.1.6) and T is a homomorphism of associative algebras. Furthermore, $T(V) = \{T(v) | v \in V\} \subset A$ is an associative subalgebra of $(A, *)$ and there is an induced dendriform algebra structure on $T(V)$ given by

$$T(u) \succ T(v) = T(u \succ v), \ \ T(u) \prec T(v) = T(u \prec v), \ \ \forall\, u, v \in V. \tag{2.1.8}$$

Moreover, its corresponding associated associative algebra structure on $T(V)$ given by equation (2.1.6) is just the associative subalgebra structure of $(A, *)$ and T is a homomorphism of dendriform algebras.

Corollary 2.1.9. ([Ag2]) *Let $(A, *)$ be an associative algebra and R be a Rota-Baxter operator (of weight zero) on A. Then there exists a dendriform algebra structure on A given by*

$$x \succ y = R(x) * y, \ \ x \prec y = x * R(y), \ \ \forall x, y \in A. \tag{2.1.9}$$

Corollary 2.1.10. ([NBG]) *Let $(A, *)$ be an associative algebra. There is a compatible dendriform algebra structure $(\succ, \prec)$ on $(A, *)$ if and only if there exists an invertible $\mathcal{O}$-operator of $(A, *)$.*

In fact, if T is an invertible $\mathcal{O}$-operator associated to a bimodule (l, r, V), then the compatible dendriform algebra structure on $(A, *)$ is given by

$$x \succ y = T(l(x)T^{-1}(y)), \ \ x \prec y = T(r(y)T^{-1}(x)), \ \ \forall x, y \in A. \tag{2.1.10}$$

Conversely, let $(A, \succ, \prec)$ be a dendriform algebra and $(A, *)$ be the associated associative algebra. Then the identity map $id : A \to A$ is an $\mathcal{O}$-operator of $(A, *)$ associated to the bimodule $(L_{\succ}, R_{\prec})$.

2.2. **$\mathcal{O}$-operators of associative algebras and associative Yang-Baxter equation.**

Definition 2.2.1. Let $(A,*)$ be an associative algebra and $r\in A\otimes A$. r is called a solution of *associative Yang-Baxter equation* in A if r satisfies

$$r_{12}*r_{13}+r_{13}*r_{23}-r_{23}*r_{12}=0. \tag{2.2.1}$$

Remark 2.2.2. The associative Yang-Baxter equation was introduced by Aguiar ([Ag1-2]) with the following form to construct an infinitesimal bialgebra:

$$r_{13}*r_{12}-r_{12}*r_{23}+r_{23}*r_{13}=0. \tag{2.2.2}$$

The form (2.2.1) of associative Yang-Baxter equation is exactly equation (2.2.2) in the opposite algebra ([Ag3]) and it was introduced in [Bai3] to construct a symmetric Frobenius algebra (see the end of this subsection). In particular, when r is skew-symmetric, it is obvious that equation (2.2.1) is just equation (2.2.2) under the operation $\sigma_{13}(x\otimes y\otimes z)=z\otimes y\otimes x$ for any $x,y,z\in A$.

Let A be a vector space. For any $r\in A\otimes A$, r can be regarded as a map from A^* to A in the following way:

$$\langle u^*\otimes v^*,r\rangle=\langle u^*,r(v^*)\rangle,\ \ \forall\, u^*,v^*\in A^*. \tag{2.2.3}$$

Proposition 2.2.3. ([Bai3]) *Let $(A,*)$ be an associative algebra and $r\in A\otimes A$. Then r is a skew-symmetric solution of associative Yang-Baxter equation in A if and only if r is an $\mathcal{O}$-operator of the associative algebra $(A,*)$ associated to the bimodule (R_*^*,L_*^*), that is, r satisfies*

$$r(a^*)*r(b^*)=r(R_*^*(r(a^*))b^*+L_*^*(r(b^*))a^*),\ \ \forall\, a^*,b^*\in A^*. \tag{2.2.4}$$

In fact, there is a more general relation between $\mathcal{O}$-operators of associative algebras and associative Yang-Baxter equation. Let $\sigma: A\otimes A\to A\otimes A$ be the exchange operator defined as

$$\sigma(x\otimes y)=y\otimes x,\ \ \forall x,y\in A. \tag{2.2.5}$$

Furthermore, let V_1,V_2 be two vector spaces and $T:V_1\to V_2$ be a linear map. Then T can be identified as an element in $V_2\otimes V_1^*$ by $\langle T,v_2^*\otimes v_1\rangle=\langle T(v_1),v_2^*\rangle$ for any $v_1\in V_1, v_2^*\in V_2^*$. Moreover, since V_2,V_1^* are the subspaces of $V_2\oplus V_1^*$, any linear map $T:V_1\to V_2$ is obviously an element in the vector space $(V_2\oplus V_1^*)\otimes(V_2\oplus V_1^*)$ (also see the proof of Theorem 3.3.5).

Theorem 2.2.4. ([BGN]) *Let $(A,*)$ be an associative algebra. Let (l,r,V) be a bimodule of A and (r^*,l^*,V^*) be the (dual) bimodule given in Example 2.1.2. Let $T:V\to A$ be a linear map identified as an element in $A\otimes V^*$ which is in the underlying vector space of $(A\ltimes_{r^*,l^*}V^*)\otimes(A\ltimes_{r^*,l^*}V^*)$. Then $r=T-\sigma(T)$ is a skew-symmetric solution of associative Yang-Baxter equation in the associative algebra $A\ltimes_{r^*,l^*}V^*$ if and only if T is an $\mathcal{O}$-operator of the associative algebra $(A,*)$ associated to the bimodule (l,r,V).*

Definition 2.2.5. Let $(A,*)$ be an associative algebra. A bilinear form $\mathcal{B}(\ ,\ )$ on A is *invariant* (or a *trace form*) if

$$\mathcal{B}(x*y,z)=\mathcal{B}(x,y*z),\ \ \forall\, x,y\in A. \tag{2.2.6}$$

A *Connes cocycle* of A is a skew-symmetric bilinear form ω satisfying

$$\omega(x*y,z)+\omega(y*z,x)+\omega(z*x,y)=0,\ \ \forall\ x,y,z\in A. \tag{2.2.7}$$

Theorem 2.2.6. ([Ag3, Proposition 2.1]) *Let A be an associative algebra and $r\in A\otimes A$. Suppose that r is skew-symmetric and nondegenerate. Then r is a solution of associative Yang-Baxter equation in A if and only if the inverse of the isomorphism $A^*\rightarrow A$ induced by r, regarded as a bilinear form ω on A, is a Connes cocycle of A. That is, $\omega(x,y)=\langle r^{-1}x,y\rangle$ for any $x,y\in A$.*

Corollary 2.2.7. ([Bai3]) *Let $(A,*)$ be an associative algebra with a nondegenerate Connes cocycle ω. Then there exists a compatible dendriform algebra structure $\succ,\prec$ on $(A,*)$ given by*

$$\omega(x\succ y,z)=\omega(y,z*x),\ \ \omega(x\prec y,z)=\omega(x,y*z),\ \ \forall x,y,z\in A. \tag{2.2.8}$$

In fact, the above dendriform algebra structure is obtained by an invertible $\mathcal{O}$-operator T associated to the bimodule (R^*,L^*), where $T:A^*\rightarrow A$ satisfies $\langle T^{-1}(x),y\rangle=\omega(x,y)$ for any $x,y\in A$.

Corollary 2.2.8. ([NBG]) *Let $(A,\succ,\prec)$ be a dendriform algebra. Then*

$$r=\sum_i^n(e_i\otimes e_i^*-e_i^*\otimes e_i) \tag{2.2.9}$$

is a solution of associative Yang-Baxter equation in the associative algebra $A\ltimes_{R_{\prec}^*,L_{\succ}^*}A^*$, where $\{e_1,\cdots,e_n\}$ is a basis of A and $\{e_1^*,\cdots,e_n^*\}$ is its dual basis. Moreover, there is a natural Connes cocycle ω on the associative algebra $A\ltimes_{R_{\prec}^*,L_{\succ}^*}A^*$ induced by $r^{-1}:A\oplus A^*\rightarrow(A\oplus A^*)^*$, which is given by

$$\omega(x+a^*,y+b^*)=-\langle x,b^*\rangle+\langle a^*,y\rangle,\ \ \forall x,y\in A,\ \ a^*,b^*\in A^*. \tag{2.2.10}$$

Remark 2.2.9. The above result can be partly reformulated from [Ag 3, Proposition 5.8].

At the end of this subsection, we briefly introduce one of the motivations to study the associative Yang-Baxter equation (the other motivations can be found in [Ag1-3]), which can be regarded as a background and explain why it is an associative analogue of the classical Yang-Baxter equation in a Lie algebra.

Recall that a (symmetric) Frobenius algebra $(A,\mathcal{B})$ is an associative algebra A with a nondegenerate (symmetric) invariant bilinear form $\mathcal{B}(\ ,\ )$. The study of Frobenius algebras plays an important role in many topics in mathematics and mathematical physics ([Ko], [RFFS], etc.). Motivated by the study of Lie bialgebras and Manin triple ([Dr], [CP]), it is natural to consider the following construction. Let $(A,*_A)$ be an associative algebra and suppose that there is another associative algebra structure $*_{A^*}$ on its dual space A^*. If there is an associative algebra structure on the direct sum $A\oplus A^*$ of the underlying vector spaces of A and A^* such that both A and A^* are subalgebras and the natural bilinear form on $A\oplus A^*$ given by

$$\mathcal{B}(x+a^*,y+b^*)=\langle x,b^*\rangle+\langle a^*,y\rangle,\ \ \forall\ a^*,b^*\in A^*,\ x,y\in A. \tag{2.2.11}$$

is invariant on $A \oplus A^*$, then it is called a *double construction of Frobenius algebra associated to* $(A, *_A)$ *and* $(A^*, *_{A^*})$ and we denote it by $(A \bowtie A^*, \mathcal{B})$.

For a linear map $\phi : V_1 \to V_2$, we denote the dual (linear) map by $\phi^* : V_2^* \to V_1^*$ defined as

$$\langle v, \phi^*(u^*)\rangle = \langle \phi(v), u^*\rangle, \quad \forall v \in V_1, u^* \in V_2. \tag{2.2.12}$$

Note that ρ^* given by equation (1.5) is different with the above notation if $gl(V)$ is regarded as a vector space, too.

Definition 2.2.10. Let $(A, *)$ be an associative algebra. An *antisymmetric infinitesimal bialgebra structure on* A is a linear map $\alpha : A \to A \otimes A$ such that

(a) $\alpha^* : A^* \otimes A^* \to A^*$ defines an associative algebra structure on A^*;

(b) α satisfies the following two equations

$$\alpha(x * y) = (1 \otimes L_*(x))\alpha(y) + (R_*(y) \otimes 1)\alpha(y); \tag{2.2.13}$$

$$(L_*(y) \otimes 1 - 1 \otimes R_*(y))\alpha(x) + \sigma[(L_*(x) \otimes 1 - 1 \otimes R_*(x))\alpha(y)] = 0, \quad \forall x, y \in A. \tag{2.2.14}$$

We denote it by (A, α).

Remark 2.2.11. Although the notion of antisymmetric infinitesimal bialgebra was introduced in [Bai3], in fact, it is the same structure in the notions of associative D-bialgebra in [Z] and balanced infinitesimal bialgebra (in the opposite algebra) in [Ag3].

Theorem 2.2.12. ([Z], [Bai3]) *Let $(A, *_A)$ be an associative algebra. Suppose there is another associative algebra structure "$*_{A^*}$" on its dual space A^* given by a linear map $\alpha^* : A^* \otimes A^* \to A^*$. Then there exists a double construction of Frobenius algebra associated to $(A, *_A)$ and $(A, *_{A^*})$ if and only if (A, α) is an antisymmetric infinitesimal bialgebra. Moreover, every double construction of Frobenius algebra can be obtained from the above way.*

The associative algebra structure on $A \oplus A^*$ is given by (for any $x, y \in A$ and $a^*, b^* \in A^*$)

$$(x+a^*)*(y+b^*) = x *_A y + R^*_{*_{A^*}}(a^*)y + L^*_{*_{A^*}}(b^*)x + a^* *_{A^*} b^* + R^*_{*_A}(x)b^* + L^*_{*_A}(y)a^*. \tag{2.2.15}$$

Proposition 2.2.13. ([Ag3], [Bai3]) *Let $(A, *)$ be an associative algebra and $r \in A \otimes A$. If r is a skew-symmetric solution of the associative Yang-Baxter equation in A, then the linear map α defined by*

$$\alpha(x) = (1 \otimes L(x) - R(x) \otimes 1)r, \quad \forall x \in A, \tag{2.2.16}$$

induces an associative algebra structure on A^* such that (A, α) is an antisymmetric infinitesimal bialgebra.

2.3. **$\mathcal{O}$-operators of associative algebras and double constructions of Connes cocycles.** Motivated by the study of Lie bialgebras ([Dr], [CP]) and antisymmetric infinitesimal bialgebras in the previous subsection, it is natural to consider the following constructions of the nondegenerate Connes cocycles of associative algebras (a similar study on the nondegenerate skew-symmetric 2-cocycles of Lie algebras has been given in [Bai2]).

Let $(A, *_A)$ be an associative algebra and suppose that there is another associative algebra structure $*_{A^*}$ on its dual space A^*. If there is an associative algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that both A and A^* are subalgebras and the skew-symmetric bilinear form on $A \oplus A^*$ given by equation (2.2.10) is a Connes cocycle on $A \oplus A^*$, then it is called a *double construction of Connes cocycle associated to* $(A, *_A)$ *and* $(A^*, *_{A^*})$ and we denote it by $(T(A) = A \bowtie A^*, \omega)$.

Definition 2.3.1. Let A be a vector space. A *dendriform D-bialgebra structure on* A is a set of linear maps $\alpha_\prec, \alpha_\succ, \beta_\prec, \beta_\succ$ such that $\alpha_\prec, \alpha_\succ : A \to A \otimes A$, $\beta_\prec, \beta_\succ : A^* \to A^* \otimes A^*$ and

(a) $(\alpha^*_\prec, \alpha^*_\succ) : A^* \otimes A^* \to A^*$ defines a dendriform algebra structure $(\succ_{A^*}, \prec_{A^*})$ on A^*;

(b) $(\beta^*_\prec, \beta^*_\succ) : A \otimes A \to A$ defines a dendriform algebra structure $(\succ_A, \prec_A)$ on A;

(c) The following equations are satisfied(for any $x, y \in A$ and $a, b \in A^*$).

(1) $\alpha_\prec(x *_A y) = (1 \otimes L_{\prec_A}(x))\alpha_\prec(y) + (R_A(y) \otimes 1)\alpha_\prec(x);$ (2.3.1)

(2) $\alpha_\succ(x *_A y) = (1 \otimes L_A(x))\alpha_\succ(y) + (R_{\prec_A}(y) \otimes 1)\alpha_\succ(x);$ (2.3.2)

(3) $\beta_\prec(a *_{A^*} b) = (1 \otimes L_{\prec_{A^*}}(a))\beta_\prec(b) + (R_{A^*}(b) \otimes 1)\beta_\prec(a);$ (2.3.3)

(4) $\beta_\succ(a *_{A^*} b) = (1 \otimes L_{A^*}(a))\beta_\succ(b) + (R_{\prec_{A^*}}(b) \otimes 1)\beta_\succ(a);$ (2.3.4)

(5) $(L_A(x) \otimes 1 - 1 \otimes R_{\prec_A}(x))\alpha_\prec(y) + \sigma[(L_{\succ_A}(y) \otimes -1 \otimes R_A(y))\alpha_\succ(x)] = 0;$ (2.3.5)

(6) $(L_{A^*}(a) \otimes 1 - 1 \otimes R_{\prec_{A^*}}(a))\beta_\prec(b) + \sigma[(L_{\succ_{A^*}}(b) \otimes -1 \otimes R_{A^*}(b))\beta_\succ(a)] = 0,$ (2.3.6)

where $L_A = L_{\succ_A} + L_{\prec_A}$, $R_A = R_{\succ_A} + R_{\prec_A}, L_{A^*} = L_{\succ_{A^*}} + L_{\prec_{A^*}}$, $R_{A^*} = R_{\succ_{A^*}} + R_{\prec_{A^*}}$.

We also denote this dendriform D-bialgebra by $(A, A^*, \alpha_\succ, \alpha_\prec, \beta_\succ, \beta_\prec)$ or simply (A, A^*).

Theorem 2.3.2. ([Bai3]) *Let $(A, \succ_A, \prec_A)$ be a dendriform algebra whose products are given by two linear maps $\beta^*_\succ, \beta^*_\prec : A \otimes A \to A$ and $(A, *_A)$ be the associated associative algebra. Suppose there is another dendriform algebra structure " $\succ_{A^*}, \prec_{A^*}$ " on its dual space A^* given by two linear maps $\alpha^*_\succ, \alpha^*_\prec : A^* \otimes A^* \to A^*$ and $(A^*, *_{A^*})$ is the associated associative algebra. Then there exists a double construction of Connes cocycle associated to $(A, *_A)$ and $(A^*, *_{A^*})$ if and only if $(A, A^*, \alpha_\succ, \alpha_\prec, \beta_\succ, \beta_\prec)$ is a dendriform D-bialgebra. Moreover, every double construction of Connes cocycle can be obtained from the above way.*

The associative algebra structure on $A \oplus A^*$ is given by (for any $x, y \in A$ and $a^*, b^* \in A^*$)

$$(x + a^*) * (y + b^*) = x *_A y + R^*_{\prec_{A^*}}(a^*)y + L^*_{\succ_{A^*}}(b^*)x + a^* *_{A^*} b^* + R^*_{\prec_A}(x)b^* + L^*_{\succ_A}(y)a^*. \quad (2.3.7)$$

Proposition 2.3.3. ([Bai3]) *Let $(A, \succ, \prec)$ be a dendriform algebra and $(A, *)$ be the associated associative algebra. Let $r \in A \otimes A$ and the maps $\alpha_\succ, \alpha_\prec$ be defined by*

$$\alpha_\succ(x) = (-1 \otimes L_*(x) + R_\prec(x) \otimes 1)r; \quad (2.3.8)$$

$$\alpha_\prec(x) = (1 \otimes L_\succ(x) - R_*(x) \otimes 1)r, \quad (2.3.9)$$

for any $x \in A$. Suppose r is symmetric and r satisfies

$$r_{12} * r_{13} = r_{13} \prec r_{23} + r_{23} \succ r_{12}. \quad (2.3.10)$$

Then the maps $\alpha_\succ, \alpha_\prec$ induce a dendriform algebra structure on A^* such that (A, A^*) is a dendriform D-bialgebra.

Definition 2.3.4. Let $(A, \succ, \prec)$ be a dendriform algebra and $r \in A \otimes A$. Equation (2.3.10) is called *D-equation in* $(A, \succ, \prec)$.

Proposition 2.3.5. ([Bai3]) *Let $(A, \succ, \prec)$ be a dendriform algebra and $(A, *)$ be the associated associative algebra. Let $r \in A \otimes A$. Then r is a symmetric solution of D-equation in the dendriform algebra $(A, \succ, \prec)$ if and only if r is an $\mathcal{O}$-operator of the associative algebra $(A, *)$ associated to the bimodule $(R^*_{\prec}, L^*_{\succ})$.*

3. $\mathcal{O}$-operators of dendriform algebras

3.1. Bimodules of dendriform algebras.

Definition 3.1.1. ([Ag4]) Let $(A, \succ, \prec)$ be a dendriform algebra and V be a vector space. Let $l_\succ, r_\succ, l_\prec, r_\prec : A \to gl(V)$ be four linear maps. V (or $(l_\succ, r_\succ, l_\prec, r_\prec)$, or $(l_\succ, r_\succ, l_\prec, r_\prec, V)$) is called a *bimodule* of A if the following equations hold (for any $x, y \in A$):

$$l_\prec(x \prec y) = l_\prec(x)l_\prec(y) + l_\prec(x)l_\succ(y); \tag{3.1.1}$$

$$r_\prec(x)l_\prec(y) = l_\prec(y)r_\prec(x) + l_\prec(y)r_\succ(x); \tag{3.1.2}$$

$$r_\prec(x)r_\prec(y) = r_\prec(y \prec x) + r_\prec(y \succ x); \tag{3.1.3}$$

$$l_\prec(x \succ y) = l_\succ(x)l_\prec(y); \tag{3.1.4}$$

$$r_\prec(x)l_\succ(y) = l_\succ(y)r_\prec(x); \tag{3.1.5}$$

$$r_\prec(x)r_\succ(y) = r_\succ(y \prec x); \tag{3.1.6}$$

$$l_\succ(x \prec y) + l_\succ(x \succ y) = l_\succ(x)l_\succ(y); \tag{3.1.7}$$

$$r_\succ(x)l_\prec(y) + r_\succ(x)l_\succ(y) = l_\succ(y)r_\succ(x); \tag{3.1.8}$$

$$r_\succ(x)r_\succ(y) + r_\succ(x)r_\prec(y) = r_\succ(y \succ x). \tag{3.1.9}$$

According to [Sc], $(l_\succ, r_\succ, l_\prec, r_\prec, V)$ is a bimodule of a dendriform algebra $(A, \succ, \prec)$ if and only if there exists a dendriform algebra structure on the direct sum $A \oplus V$ of the underlying vector spaces of A and V given by ($\forall x, y \in A, u, v \in V$)

$$(x+u) \succ (y+v) = x \succ y + l_\succ(x)v + r_\succ(y)u, \;\; (x+u) \prec (y+v) = x \prec y + l_\prec(x)v + r_\prec(y)u. \tag{3.1.10}$$

We denote it by $A \ltimes_{l_\succ, r_\succ, l_\prec, r_\prec} V$.

Proposition 3.1.2. ([Bai3]) *Let $(l_\succ, r_\succ, l_\prec, r_\prec,\ V)$ be a bimodule of a dendriform algebra $(A, \succ, \prec)$. Let $(A, *)$ be the associated associative algebra.*

(1) Both $(l_\succ, r_\prec, V)$ and $(l_\succ + l_\prec, r_\succ + r_\prec, V)$ are bimodules of $(A, *)$.

(2) For any bimodule (l, r, V) of $(A, *)$, $(l, 0, 0, r, V)$ is a bimodule of $(A, \succ, \prec)$.

(3) Both $(l_\succ + l_\prec, 0, 0, r_\succ + r_\prec, V)$ and $(l_\succ, 0, 0, r_\prec, V)$ are bimodules of $(A, \succ, \prec)$.

(4) The dendriform algebras $A \ltimes_{l_\succ, r_\succ, l_\prec, r_\prec} V$ and $A \ltimes_{l_\succ + l_\prec, 0, 0, r_\succ + r_\prec} V$ have the same associated associative algebra $A \ltimes_{l_\succ + l_\prec, r_\succ + r_\prec} V$.

Proposition 3.1.3. ([Bai3]) *Let $(A,\succ,\prec)$ be a dendriform algebra and $(l_\succ,r_\succ,l_\prec,r_\prec,V)$ be a bimodule. Then $(r_\succ^*+r_\prec^*,-l_\prec^*,-r_\succ^*,l_\succ^*+l_\prec^*,V^*)$ is a bimodule of $(A,\succ,\prec)$. We call it the dual bimodule of $(l_\succ,r_\succ,l_\prec,r_\prec,V)$.*

Corollary 3.1.4. ([Bai3]) *Let $(A,\succ,\prec)$ be a dendriform algebra and $(l_\succ,r_\succ,l_\prec,r_\prec,V)$ be a bimodule. Let $(A,*)$ be the associated associative algebra.*

(1) Both $(r_\succ^*+r_\prec^*,0,0,l_\succ^*+l_\prec^*,V^*)$ and $(r_\prec^*,0,0,l_\succ^*,V^*)$ are bimodules of $(A,\succ,\prec)$.

(2) Both $(r_\succ^*+r_\prec^*,l_\succ^*+l_\prec^*,V^*)$ and $(r_\prec^*,l_\succ^*,V^*)$ are bimodules of $(A,*)$.

Example 3.1.5. Let $(A,\succ,\prec)$ be a dendriform algebra. Then

$$(L_\succ,R_\succ,L_\prec,R_\prec,A),\quad (L_\succ,0,0,R_\prec,A)\ \text{ and }\ (L_\succ+L_\prec,0,0,R_\succ+R_\prec,A)$$

are bimodules of $(A,\prec,\succ)$ and the first one is called the regular bimodule of $(A,\succ,\prec)$. On the other hand,

$$(R_\succ^*+R_\prec^*,-L_\prec^*,-R_\succ^*,L_\succ^*+L_\prec^*,A^*),\ \ (R_\prec^*,0,0,L_\succ^*,A^*)\ \text{ and }\ (R_\succ^*+R_\prec^*,0,0,L_\succ^*+L_\prec^*,A^*)$$

are bimodules of $(A,\succ,\prec)$, too.

3.2. **Bilinear forms on dendriform algebras and D-equation.** In fact, a double construction of Connes cocycle on an associative algebra $(A,*)$ is equivalent to a double construction of certain nondegenerate bilinear form on its compatible dendriform algebra.

Definition 3.2.1. Let $(A,\succ,\prec)$ be a dendriform algebra. A skew-symmetric bilinear form ω on A is called *invariant* if ω satisfies (for any $x,y,z\in A$)

$$\omega(x\succ y,z)=\omega(y,z\succ x+z\prec x), \tag{3.2.1}$$

$$\omega(x\prec y,z)=\omega(x,y\succ z+y\prec z). \tag{3.2.2}$$

Proposition 3.2.2. *Let $(A,\succ,\prec)$ be a dendriform algebra with a skew-symmetric bilinear form ω.*

(1) ω is invariant if and only if ω satisfies equation (3.2.1) and

$$\omega(x\succ y,z)+\omega(x,y\prec z)=0,\ \ \forall\, x,y,z\in A. \tag{3.2.3}$$

(2) ω is invariant if and only if ω satisfies equations (3.2.2) and (3.2.3).

(3) ω is invariant if and only if ω satisfies equation (3.2.1) and

$$\omega(x\succ y,z)+\omega(y\succ z,x)+\omega(z\succ x,y)=0,\ \ \forall x,y,z\in A. \tag{3.2.4}$$

(4) ω is invariant if and only if ω satisfies equation (3.2.2) and

$$\omega(x\prec y,z)+\omega(y\prec z,x)+\omega(z\prec x,y)=0,\ \ \forall x,y,z\in A. \tag{3.2.5}$$

(5) If ω is invariant, then ω is a Connes cocycle of the associated associative algebra $(A,*)$.

Proof. (1) Let ω be a skew-symmetric bilinear form satisfying equation (3.2.1). If ω satisfies equation (3.2.2), then

$$\omega(x \succ y, z) = \omega(y, z \succ x + z \prec x) = \omega(y \prec z, x) = -\omega(x, y \prec z), \quad \forall x, y, z \in A.$$

Conversely, if ω satisfies equation (3.2.3), then

$$\omega(x \prec y, z) = \omega(z \succ x, y) = \omega(x, y \succ z + y \prec z), \quad \forall x, y, z \in A.$$

By a similar discussion as in (1), the conclusion (2) holds.

(3) If the skew-symmetric bilinear form ω satisfies equation (3.2.1), then

$$\omega(x \succ y, z) = \omega(y, z \succ x + z \prec x) = -\omega(z \succ x, y) - \omega(z \prec x, y). \ \forall x, y, z \in A.$$

Therefore equation (3.2.3) holds if and only if equation (3.2.4) holds. By (1), the conclusion (3) follows.

The conclusion (4) follows by a similar discussion as in (3).

(5) follows immediately from the sum of equations (3.2.1) and (3.2.2). □

By Corollary 2.2.7 and the conclusion (5) in Proposition 3.2.2, we have the following result.

Corollary 3.2.3. *Let $(A, *)$ be an associative algebra and ω be a nondegenerate skew-symmetric bilinear form. Then ω is a Connes cocycle of $(A, *)$ if and only if ω is invariant on the compatible dendriform algebra given by equation (2.2.8).*

Let $(A, \succ_A, \prec_A)$ be a dendriform algebra and suppose that there is another dendriform algebra structure $\succ_{A^*}, \prec_{A^*}$ on its dual space A^*. If there is a dendriform algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that both A and A^* are subalgebras and the skew-symmetric bilinear form on $A \oplus A^*$ given by equation (2.2.10) is invariant on $A \oplus A^*$, then it is called a *double construction of a dendriform algebra with a nondegenerate invariant bilinear form associated to* $(A, \succ_A, \prec_A)$ *and* $(A^*, \succ_{A^*}, \prec_{A^*})$.

By Corollary 3.2.3 and Theorem 2.3.2, we have the following conclusion.

Corollary 3.2.4. *Let $(A, \succ_A, \prec_A)$ be a dendriform algebra whose products are given by two linear maps $\beta^*_{\succ}, \beta^*_{\prec} : A \otimes A \to A$. Suppose there is another dendriform algebra structure " $\succ_{A^*}$, $\prec_{A^*}$ " on its dual space A^* given by two linear maps $\alpha^*_{\succ}, \alpha^*_{\prec} : A^* \otimes A^* \to A^*$. Then there exists a double construction of a dendriform algebra with a nondegenerate invariant bilinear form associated to $(A, \succ_A, \prec_A)$ and $(A^*, \succ_{A^*}, \prec_{A^*})$ if and only if $(A, A^*, \alpha_{\succ}, \alpha_{\prec}, \beta_{\succ}, \beta_{\prec})$ is a dendriform D-bialgebra. Moreover, every double construction of a dendriform algebra with a nondegenerate invariant bilinear form can be obtained from the above way.*

The dendriform algebra structure on $A \oplus A^*$ is given by (for any $x, y \in A$ and $a^*, b^* \in A^*$)

$$(x+a^*) \succ (y+b^*) = x \succ_A y + R^*_{A^*}(a^*)y - L^*_{\prec_{A^*}}(b^*)x + a^* \succ_{A^*} b^* + R^*_A(x)b^* - L^*_{\prec_A}(y)a^*, \quad (3.2.6)$$

$$(x+a^*) \prec (y+b^*) = x \succ_A y - R^*_{\succ_{A^*}}(a^*)y + L^*_{A^*}(b^*)x + a^* \prec_{A^*} b^* - R^*_{\succ_A}(x)b^* + L^*_A(y)a^*, \quad (3.2.7)$$

where $R^*_{A^*} = R^*_{\prec_{A^*}} + R^*_{\succ_{A^*}}, R^*_A = R^*_{\prec_A} + R^*_{\succ_A}, L^*_{A^*} = L^*_{\prec_{A^*}} + R^*_{\succ_{A^*}}, L^*_A = L^*_{\prec_A} + R^*_{\succ_A}$.

Corollary 3.2.5. *Let $(A,\succ,\prec)$ be a dendriform algebra and $r\in A\otimes A$. Let the linear maps $\alpha_\succ,\alpha_\prec$ be defined by equations (2.3.8) and (2.3.9). If r is a symmetric solution of D-equation in A, then the maps $\alpha_\succ,\alpha_\prec$ induce a dendriform algebra structure on A^* such that there is a double construction of a dendriform algebra with a nondegenerate invariant bilinear form associated to $(A,\succ_A,\prec_A)$ and $(A^*,\succ_{A^*},\prec_{A^*})$.*

Remark 3.2.6. In the above sense, the D-equation in a dendriform algebra is just an analogue of the classical Yang-Baxter equation in a Lie algebra.

Next, we consider the symmetric bilinear forms on a dendriform algebra.

Theorem 3.2.7. ([Bai3]) *Let $(A,\succ,\prec)$ be a dendriform algebra and $r\in A\otimes A$. Suppose that r is symmetric and nondegenerate. Then r is a solution of D-equation in A if and only if the inverse of the isomorphism $A^*\to A$ induced by r, regarded as a bilinear form $\mathcal{B}$ on A (that is, $\mathcal{B}(x,y)=\langle r^{-1}x,y\rangle$ for any $x,y\in A$) satisfies*

$$\mathcal{B}(x*y,z)=\mathcal{B}(y,z\prec x)+\mathcal{B}(x,y\succ z),\ \ \forall\, x,y,z\in A. \tag{3.2.8}$$

Definition 3.2.8. Let $(A,\succ,\prec)$ be a dendriform algebra. A symmetric bilinear form $\mathcal{B}$ on A is called *2-cocycle* of $(A,\succ,\prec)$ if $\mathcal{B}$ satisfies equation (3.2.8).

3.3. $\mathcal{O}$-operators of dendriform algebras and D-equation.

Definition 3.3.1. Let $(A,\succ,\prec)$ be a dendriform algebra and $(l_\succ,r_\succ,l_\prec,r_\prec,V)$ be a bimodule. A linear map $T:V\to A$ is called an *$\mathcal{O}$-operator of $(A,\succ,\prec)$ associated to $(l_\succ,r_\succ,l_\prec,r_\prec,V)$* if T satisfies (for any $u,v\in V$)

$$T(u)\succ T(v)=T(l_\succ(T(u))v+r_\succ(T(v)u)),T(u)\prec T(v)=T(l_\prec(T(u))v+r_\prec(T(v)u)). \tag{3.3.1}$$

The following result is obvious.

Corollary 3.3.2. *Let $(l_\succ,r_\succ,l_\prec,r_\prec,V)$ be a bimodule of a dendriform algebra $(A,\succ,\prec)$. Let $(A,*)$ be the associated associative algebra. If T is an $\mathcal{O}$-operator of $(A,\succ,\prec)$ associated to $(l_\succ,r_\succ,l_\prec,r_\prec,V)$, then T is an $\mathcal{O}$-operator of $(A,*)$ associated to $(l_\succ+l_\prec,r_\succ+r_\prec,V)$.*

Theorem 3.3.3. *Let $(A,\succ,\prec)$ be a dendriform algebra and $(A,*)$ be the associated associative algebra. Let $r\in A\otimes A$ be symmetric. Then the following conditions are equivalent.*

(1) r is a solution of D-equation in $(A,\succ,\prec)$.

(2) r is an $\mathcal{O}$-operator of $(A,*)$ associated to $(R^*_\prec,L^*_\succ)$.

(3) r satisfies

$$r(a^*)\succ r(b^*)=r(R^*_*(r(a^*))b^*-L^*_\prec(r(b^*))a^*),\ \ \forall a^*,b^*\in A^*. \tag{3.3.2}$$

(4) r satisfies

$$r(a^*)\prec r(b^*)=r(-R^*_\succ(r(a^*))b^*+L^*_*(r(b^*))a^*),\ \ \forall a^*,b^*\in A^*. \tag{3.3.3}$$

Proof. The fact that (1) is equivalent to (2) follows from Proposition 2.3.5. Let $\{e_1,\cdots,e_n\}$ be a basis of A and $\{e_1^*,\cdots,e_n^*\}$ be its dual basis. Suppose that

$$e_i\succ e_j=\sum_k a_{ij}^k e_k,\ \ e_i\prec e_j=\sum_k b_{ij}^k e_k,\ \ r=\sum_{i,j} r_{ij}e_i\otimes e_j,\ \ r_{ij}=r_{ji}.$$

Hence $r(e_i^*)=\sum_k r_{ik}e_k$. Then r is a solution of D-equation in A if and only if (for any m,t,p)

$$\sum_{i,k}[r_{it}r_{kp}(a_{ik}^m+b_{ik}^m)-r_{mi}r_{tk}b_{ik}^p-r_{ip}r_{mk}a_{ik}^t]=0.$$

The left-hand side of the above equation is precisely the coefficient of e_t in

$$-r(e_p^*)\succ r(e_m^*)+r(R_*^*(r(e_p^*))e_m^*-L_\prec^*(r(e_m^*))e_p^*),$$

and the coefficient of e_p in

$$-r(e_m^*)\prec r(e_t^*)+r(-R_\succ^*(r(e_m^*))e_t^*+L_*^*(r(e_t^*))e_m^*).$$

Therefore the conclusion follows. $\square$

Corollary 3.3.4. *Let $(A,\succ,\prec)$ be a dendriform algebra and $r\in A\otimes A$ be symmetric. Then r is a solution of D-equation in $(A,\succ,\prec)$ if and only if r is an $\mathcal{O}$-operator of $(A,\succ,\prec)$ associated to $(R_\succ^*+R_\prec^*,-L_\prec^*,-R_\succ^*,L_\succ^*+L_\prec^*,A^*)$.*

Theorem 3.3.5. *Let $(A,\succ,\prec)$ be a dendriform algebra. Let $(l_\succ,r_\succ,l_\prec,r_\prec,V)$ be a bimodule and $(r_\succ^*+r_\prec^*,-l_\prec^*,-r_\succ^*,l_\succ^*+l_\prec^*,V^*)$ be the dual bimodule given by Proposition 3.1.3. Let $T:V\to A$ be a linear map identified as an element in $A\otimes V^*$ which is in the underlying vector space of $(A\ltimes_{r_\succ^*+r_\prec^*,-l_\prec^*,-r_\succ^*,l_\succ^*+l_\prec^*}V^*)\otimes(A\ltimes_{r_\succ^*+r_\prec^*,-l_\prec^*,-r_\succ^*,l_\succ^*+l_\prec^*}V^*)$. Then $r=T+\sigma(T)$ is a symmetric solution of D-equation in the dendriform algebra $A\ltimes_{r_\succ^*+r_\prec^*,-l_\prec^*,-r_\succ^*,l_\succ^*+l_\prec^*}V^*$ if and only if T is an $\mathcal{O}$-operator of $(A,\succ,\prec)$ associated to $(l_\succ,r_\succ,l_\prec,r_\prec,V)$.*

Proof. Let $\{e_1,\cdots,e_n\}$ be a basis of A. Let $\{v_1,\cdots,v_m\}$ be a basis of V and $\{v_1^*,\cdots,v_m^*\}$ be its dual basis. Set $T(v_i)=\sum\limits_{j=1}^n a_{ij}e_j, i=1,\cdots,m$. Since $\mathrm{Hom}(V,A)\cong A\otimes V^*$ as vector spaces,

$$\begin{aligned}T&=\sum_{i=1}^m T(v_i)\otimes v_i^*=\sum_{i=1}^m\sum_{j=1}^n a_{ij}e_j\otimes v_i^*\in A\otimes V^*\\&\subset (A\ltimes_{r_\succ^*+r_\prec^*,-l_\prec^*,-r_\succ^*,l_\succ^*+l_\prec^*}V^*)\otimes(A\ltimes_{r_\succ^*+r_\prec^*,-l_\prec^*,-r_\succ^*,l_\succ^*+l_\prec^*}V^*).\end{aligned}$$

Therefore we have

$$\begin{aligned}r_{12}*r_{13}&=\sum_{i,k=1}^m\{T(v_i)*T(v_k)\otimes v_i^*\otimes v_k^*+r_\prec^*(T(v_i))v_k^*\otimes v_i^*\otimes T(v_k)\\&\qquad+l_\succ^*(T(v_k))v_i^*\otimes T(v_i)\otimes v_k^*\};\\ r_{13}\prec r_{23}&=\sum_{i,k=1}^m\{T(v_i)\otimes v_k^*\otimes(l_\succ^*+l_\prec^*)(T(v_k))v_i^*-v_i^*\otimes T(v_k)\otimes r_\succ^*(T(v_i))v_k^*\\&\qquad+v_i^*\otimes v_k^*\otimes T(v_i)\prec T(v_k)\};\end{aligned}$$

$$r_{23} \succ r_{12} = \sum_{i,k=1}^{m} \{T(v_k) \otimes (r_{\succ}^* + r_{\prec}^*)(T(v_i))v_k^* \otimes v_i^* + v_k^* \otimes T(v_i) \succ T(V_k) \otimes v_i^*$$
$$-v_k^* \otimes l_{\prec}^*(T(v_j))v_i^* \otimes T(v_i)\}.$$

Furthermore, by equation (1.5), we show that

$$l_{\succ}^*(T(v_k))v_i^* = \sum_{j=1}^{m} v_i^*(l_{\succ}(T(v_k))v_j)v_j^*, \quad r_{\succ}^*(T(v_k))v_i^* = \sum_{j=1}^{m} v_i^*(r_{\succ}(T(v_k))v_j)v_j^*.$$
$$l_{\prec}^*(T(v_k))v_i^* = \sum_{j=1}^{m} v_i^*(l_{\prec}(T(v_k))v_j)v_j^*, \quad r_{\prec}^*(T(v_k))v_i^* = \sum_{j=1}^{m} v_i^*(r_{\prec}(T(v_k))v_j)v_j^*.$$

Thus

$$\sum_{i,k=1}^{m} T(v_i) \otimes l_{\succ}^*(T(v_k))v_i^* = \sum_{i,k=1}^{m} T(v_i) \otimes [\sum_{j=1}^{m} v_i^*(l_{\succ}(T(v_k))v_j)v_j^*]$$
$$= \sum_{i,k=1}^{m}\sum_{j=1}^{m} v_j^*(l_{\succ}(T(v_k))v_i)T(v_j) \otimes v_i^* = \sum_{i,k=1}^{m} T(l_{\succ}(T(v_k))v_i) \otimes v_i^*.$$

Similarly, we have

$$\sum_{i,k=1}^{m} T(v_i) \otimes r_{\succ}^*(T(v_k))v_i^* = \sum_{i,k=1}^{m} T(r_{\succ}(T(v_k))v_i) \otimes v_i^*;$$
$$\sum_{i,k=1}^{m} T(v_i) \otimes l_{\prec}^*(T(v_k))v_i^* = \sum_{i,k=1}^{m} T(l_{\prec}(T(v_k))v_i) \otimes v_i^*;$$
$$\sum_{i,k=1}^{m} T(v_i) \otimes r_{\prec}^*(T(v_k))v_i^* = \sum_{i,k=1}^{m} T(r_{\prec}(T(v_k))v_i) \otimes v_i^*.$$

Therefore

$$r_{12} * r_{13} - r_{13} \prec r_{23} - r_{23} \succ r_{12}$$
$$= \sum_{i,k=1}^{m} \{(T(v_i) * T(v_k) - T((r_{\succ} + r_{\prec})(T(v_k))v_i) - T((l_{\succ} + l_{\prec})(T(v_i))v_k)) \otimes v_i^* \otimes v_k^*$$
$$+v_i^* \otimes (-T(v_i) \succ T(v_k) + T(r_{\succ}(T(v_k))v_i) + T(l_{\succ}(T(v_i))v_k)) \otimes v_k^*$$
$$+v_i^* \otimes v_k^* \otimes (-T(v_i) \prec T(v_k) + T(r_{\prec}(T(v_k))v_i) + T(l_{\prec}(T(v_i))v_k))\}.$$

So r is a symmetric solution of D-equation in the dendriform algebra $A \ltimes_{r_{\succ}^*+r_{\prec}^*, -l_{\prec}^*, -r_{\succ}^*, l_{\succ}^*+l_{\prec}^*} V^*$ if and only if T is an $\mathcal{O}$-operator of $(A, \succ, \prec)$ associated to $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V)$. □

Corollary 3.3.6. *Let $(A, *)$ be an associative algebra. Let (l, r, V) be a bimodule and (r^*, l^*, V^*) be the dual bimodule given in Example 2.1.2. Suppose that $T : V \to A$ is an $\mathcal{O}$-operator of $(A, *)$ associated to (l, r, V). Then $r = T + \sigma(T)$ is a symmetric solution of D-equation in the dendriform algebra $T(V) \ltimes_{r^*,0,0,l^*} V^*$, where $T(V) \subset A$ is a dendriform algebra given by equation (2.1.8) and $(r^*, 0, 0, l^*)$ is its bimodule since its associated associative algebra $T(V)$ is an associative subalgebra of A, and T can be identified as an element in $T(V) \otimes V^*$ which is in the underlying vector space of $(T(V) \ltimes_{r^*,0,0,l^*} V^*) \otimes (T(V) \ltimes_{r^*,0,0,l^*} V^*)$.*

Proof. By Theorem 2.1.8, it is obvious that $T: V \to T(V)$ is an $\mathcal{O}$-operator of $(T(V), \succ, \prec)$ associated to the bimodule $(l, 0, 0, r, V)$, where

$$T(u) \succ T(v) = T(l(T(u))v), \ \ T(u) \prec T(v) = T(r(T(v))u), \ \ \forall\, u, v \in V.$$

Hence the conclusion follows from Theorem 3.3.5 immediately. □

Remark 3.3.7. The above conclusion has appeared in [Bai3] with a direct proof. We would like to emphasis that it involves only the $\mathcal{O}$-operators of associative algebras (not the $\mathcal{O}$-operators of dendriform algebras). Therefore, as has been pointed out in [Bai3], roughly speaking, the symmetric part of an $\mathcal{O}$-operator of an associative algebra corresponds to a symmetric solution of D-equation, whereas the skew-symmetric part of an $\mathcal{O}$-operator of an associative algebra corresponds to a skew-symmetric solution of associative Yang-Baxter equation.

Corollary 3.3.8. (cf. Proposition 3.4.12) *Let $(A, \succ, \prec)$ be a dendriform algebra. Then*

$$r = \sum_{i}^{n} (e_i \otimes e_i^* + e_i^* \otimes e_i) \tag{3.3.4}$$

is a symmetric solution of D-equation in the dendriform algebra $A \ltimes_{R_{\prec}^*, 0, 0, L_{\succ}^*} A^*$, where $\{e_1, \cdots, e_n\}$ is a basis of A and $\{e_1^*, \cdots, e_n^*\}$ is its dual basis. Moreover there is a natural 2-cocycle $\mathcal{B}$ of the dendriform algebra $A \ltimes_{R_{\prec}^*, 0, 0, L_{\succ}^*} A^*$ induced by $r^{-1}: A \oplus A^* \to (A \oplus A^*)^*$, which is given by equation (2.2.11).

Proof. Since id is an $\mathcal{O}$-operator of $(A, \succ, \prec)$ associated to the bimodule $(L_{\succ}, 0, 0, R_{\prec}, A)$, r is a symmetric solution of D-equation in $A \ltimes_{R_{\prec}^*, 0, 0, L_{\succ}^*} A^*$ due to Theorem 3.3.5. Therefore the bilinear form $\mathcal{B}$ given by equation (2.2.11) is a 2-cocycle due to Theorem 3.2.7. □

3.4. $\mathcal{O}$-operators of dendriform algebras and quadri-algebras.

Definition 3.4.1. ([AL]) Let A be a vector space with four bilinear products denoted by $\searrow, \nearrow, \nwarrow$ and $\swarrow: A \otimes A \to A$. $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called a *quadri-algebra* if for any $x, y, z \in A$,

$$(x \nwarrow y) \nwarrow z = x \nwarrow (y * z), \ \ (x \nearrow y) \nwarrow z = x \nearrow (y \prec z), \ \ (x \wedge y) \nearrow z = x \nearrow (y \succ z), \tag{3.4.1}$$

$$(x \swarrow y) \nwarrow z = x \swarrow (y \wedge z), \ \ (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), \ \ (x \vee y) \nearrow z = x \searrow (y \nearrow z), \tag{3.4.2}$$

$$(x \prec y) \swarrow z = x \swarrow (y \vee z), \ \ (x \succ y) \swarrow z = x \searrow (y \swarrow z), \ \ (x * y) \searrow z = x \searrow (y \searrow z), \tag{3.4.3}$$

where

$$x \succ y = x \nearrow y + x \searrow y, x \prec y = x \nwarrow y + x \swarrow y, x \vee y = x \searrow y + x \swarrow y, x \wedge y = x \nearrow y + x \nwarrow y, \tag{3.4.4}$$

and

$$x * y = x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y = x \succ y + x \prec y = x \vee y + x \wedge y. \tag{3.4.5}$$

Proposition 3.4.2. ([AL]) *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra.*

(1) The product given by

$$x \succ y = x \nearrow y + x \searrow y, \quad x \prec y = x \nwarrow y + x \swarrow y, \quad \forall x, y \in A, \tag{3.4.6}$$

defines a dendriform algebra. $(A, \succ, \prec)$ is called the associated horizontal dendriform algebra of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ and $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called a compatible quadri-algebra structure on the horizontal dendriform algebra $(A, \succ, \prec)$.

(2) The product given by

$$x \vee y = x \searrow y + x \swarrow y, \; x \wedge y = x \nearrow y + x \nwarrow y, \quad \forall x, y \in A, \tag{3.4.7}$$

defines a dendriform algebra. $(A, \vee, \wedge)$ is called the associated vertical dendriform algebra of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ and $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called a compatible quadri-algebra structure on the (vertical) dendriform algebra $(A, \vee, \wedge)$.

(3) The product given by equation (3.4.5) defines an associative algebra. $(A, *)$ is called the associated associative algebra of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ and $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called a compatible quadri-algebra structure on the associative algebra $(A, *)$.

Proposition 3.4.3. *Let A be a vector space with four bilinear products denoted by $\searrow, \nearrow, \nwarrow$ and $\swarrow : A \otimes A \to A$.*

(1) $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is a quadri-algebra if and only if $(A, \succ, \prec)$ defined by equation (3.4.6) is a dendriform algebra and $(L_{\searrow}, R_{\nearrow}, L_{\swarrow}, R_{\nwarrow}, A)$ is a bimodule.

(2) $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is a quadri-algebra if and only if $(A, \vee, \wedge)$ defined by equation (3.4.7) is a dendriform algebra and $(L_{\searrow}, R_{\swarrow}, L_{\nearrow}, R_{\nwarrow}, A)$ is a bimodule.

Proof. The conclusions can be obtained by a direct computation or a similar proof as of Proposition 3.4.6. □

Corollary 3.4.4. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. Then $(L_{\searrow}, R_{\nwarrow}, A)$ is a bimodule of the associated associative algebra $(A, *)$.*

Proof. It follows immediately from Propositions 3.1.2 and 3.4.3. □

For brevity, we pay our main attention to the study of associated horizontal dendriform algebras. In fact, the corresponding study on the associated vertical dendriform algebras are completely similar.

Corollary 3.4.5. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra and $(A, \succ, \prec)$ be the associated horizontal dendriform algebra. Then*

$$(L_{\succ}, R_{\succ}, L_{\prec}, R_{\prec}, A), \quad (L_{\succ}, 0, 0, R_{\prec}, A), \quad (L_{\succ} + L_{\prec}, 0, 0, R_{\succ} + R_{\prec}, A);$$

$$(L_{\searrow}, R_{\nearrow}, L_{\swarrow}, R_{\nwarrow}, A), \quad (L_{\searrow}, 0, 0, R_{\nwarrow}, A) \text{ and } (L_{\vee}, 0, 0, R_{\wedge}, A)$$

are bimodules of $(A, \prec, \succ)$. On the other hand,

$$(R_{\succ}^* + R_{\prec}^*, -L_{\prec}^*, -R_{\succ}^*, L_{\succ}^* + L_{\prec}^*, A^*), \;\; (R_{\prec}^*, 0, 0, L_{\succ}^*, A^*) \;\; (R_{\succ}^* + R_{\prec}^*, 0, 0, L_{\succ}^* + L_{\prec}^*, A^*);$$

$$(R^*_{\nearrow}+R^*_{\nwarrow},-L^*_{\swarrow},-R^*_{\nearrow},L^*_{\searrow}+L^*_{\swarrow},A^*),\ \ (R^*_{\nwarrow},0,0,L^*_{\searrow},A^*)\ \text{ and }\ (R^*_{\wedge},0,0,L^*_{\vee},A^*)$$

are bimodules of $(A,\succ,\prec)$, too.

Proof. It follows immediately from Propositions 3.1.2, 3.1.3 and 3.4.3 and Example 3.1.5. □

Proposition 3.4.6. *Let $(A,\succ,\prec)$ be a dendriform algebra and $(l_\succ,r_\succ,l_\prec,r_\prec,V)$ be a bimodule. Let $T:V\to A$ be an $\mathcal{O}$-operator associated to $(l_\succ,r_\succ,l_\prec,r_\prec,V)$. Then there exists a quadri-algebra structure on V given by (for any $u,v\in V$)*

$$u\searrow v=l_\succ(T(u))v,\ u\nearrow v=r_\succ(T(v))u,\ u\swarrow v=l_\prec(T(u))v,\ u\nwarrow v=r_\prec(T(v))u. \tag{3.4.8}$$

Therefore, there exists a dendriform algebra structure on V given by equation (3.4.6) and T is a homomorphism of dendriform algebras. Furthermore, $T(V)=\{T(v)|v\in V\}\subset A$ is a dendriform subalgebra of A and there is an induced quadri-algebra structure on $T(V)$ given by

$$T(u)\searrow T(v)=T(u\searrow v),\ T(u)\nearrow T(v)=T(u\nearrow v),$$

$$T(u)\swarrow T(v)=T(u\swarrow v),\ T(u)\nwarrow T(v)=T(u\nwarrow v),\ \forall u,v\in V. \tag{3.4.9}$$

Moreover, its corresponding associated horizontal dendriform algebra structure on $T(V)$ given by equation (3.4.6) is just the dendriform subalgebra structure of $(A,\succ,\prec)$ and T is a homomorphism of quadri-algebras.

Proof. Set $l=l_\succ+l_\prec$ and $r=r_\succ+r_\prec$. For any $u,v,w\in V$, we have

$$\begin{aligned}
(u\nwarrow v)\nwarrow w &= r_\prec(T(w))(u\nwarrow v)=r_\prec(T(w))r_\prec(T(v))u\overset{(3.1.3)}{=}r_\prec(T(v)*T(w))u\\
&= u\nwarrow(l(T(v))w+r(T(w))v)=u\nwarrow(v*w);\\
(u\nearrow v)\nwarrow w &= r_\prec(T(w))(u\nwarrow v)=r_\prec(T(w))r_\succ(T(v))u\overset{(3.1.6)}{=}r_\succ(T(v)\prec T(w))u\\
&= u\nwarrow(l_\prec(T(v))w+r_\prec(T(w))v)=u\nwarrow(v\prec w);\\
(u\wedge v)\nearrow w &= r_\succ(T(w))(u\nearrow v+u\nwarrow v)=r_\succ(T(w))(r_\succ(T(v))u+r_\prec(T(v))u)\\
&\overset{(3.1.9)}{=} r_\succ(T(w)\succ T(v))u=u\nearrow(l_\succ(T(w))v+r_\succ(T(v))w)=u\nearrow(v\succ w);\\
(u\swarrow v)\nwarrow w &= r_\prec(T(w))(u\swarrow v)=r_\prec(T(w))l_\prec(T(u))v\\
&\overset{(3.1.2)}{=} l_\prec(T(u))(r_\prec(T(w))+r_\succ(T(w)))v=u\swarrow(v\wedge w);\\
(u\searrow v)\nwarrow w &= r_\prec(T(w))(u\searrow v)=r_\prec(T(w))l_\succ(T(u))v\overset{(3.1.5)}{=}l_\succ(T(u))r_\succ(T(w)))v\\
&= l_\succ(T(u))(v\nwarrow w)=u\searrow(v\nwarrow w);\\
(u\vee v)\nearrow w &= r_\succ(T(w))(l_\succ(T(u))+l_\prec(T(u))v\overset{(3.1.8)}{=}l_\succ(T(u))r_\succ(T(w))v\\
&= l_\succ(T(u))(v\nearrow w)=u\searrow(v\nearrow w);\\
(u\prec v)\swarrow w &= l_\succ(T(u\prec v))w=l_\prec(T(u)\prec T(v))w\\
&\overset{(3.1.1)}{=} l_\prec(T(u))(l_\prec(T(v))+l_\succ(T(v)))w=u\swarrow(v\vee w);
\end{aligned}$$

$$\begin{aligned}
(u \succ v) \swarrow w &= l_{\prec}(T(u \succ v))w = l_{\prec}(T(u) \succ T(v))w \overset{(3.1.4)}{=} l_{\succ}(T(u))l_{\prec}(T(v))w \\
&= l_{\succ}(T(u))(v \swarrow w) = u \searrow (u \swarrow w); \\
(u * v) \searrow w &= l_{\succ}(T(u * v))w = l_{\succ}(T(u) * T(v))w \overset{(3.1.7)}{=} l_{\succ}(T(u))l_{\succ}(T(v))w \\
&= l_{\succ}(T(u))(v \searrow w) = u \searrow (v \searrow w).
\end{aligned}$$

Therefore $(V, \searrow, \nearrow, \nwarrow, \swarrow)$ is a quadri-algebra. Furthermore the other results follow easily. $\square$

Definition 3.4.7. ([AL]) Let $(A, \succ, \prec)$ be a dendriform algebra. An $\mathcal{O}$-operator R of $(A, \succ, \prec)$ associated to the regular bimodule $(L_{\succ}, R_{\succ}, L_{\prec}, R_{\prec}, A)$ is called a *Rota-Baxter operator* on A, that is, R satisfies

$$R(x \succ y) = R(R(x) \succ y + x \succ R(y)),\ R(x \prec y) = R(R(x) \prec y + x \prec R(y)),\ \forall x, y \in A. \quad (3.4.10)$$

By Proposition 3.4.6, the following conclusion follows immediately.

Corollary 3.4.8. ([AL], Proposition 2.3) *Let $(A, \succ, \prec)$ be a dendriform algebra and R be a Rota-Baxter operator on $(A, \succ, \prec)$. Then there exists a quadri-algebra structure on A defined by (for any $x, y \in A$)*

$$x \searrow y = R(x) \succ y,\ x \nearrow y = x \succ R(y),\ x \swarrow y = R(x) \prec y,\ x \nwarrow y = x \prec R(y). \quad (3.4.11)$$

Moreover, by Corollaries 2.1.9 and 3.4.8, it is obvious that the Rota-Baxter operators on associative algebras can construct quadri-algebras as follows (also see Lemma 4.4.9).

Corollary 3.4.9. ([AL], Corollary 2.6) *Let R_1 and R_2 be a pair of commuting Rota-Baxter operators (of weight zero) on an associative algebra $(A, *)$. Then there exists a quadri-algebra structure on A defined by (for any $x, y \in A$)*

$$x \searrow y = R_1R_2(x) * y,\ x \nearrow y = R_1(x) * R_2(y),\ x \swarrow y = R_2(x) * R_1(y),\ x \nwarrow y = x * R_1R_2(y). \quad (3.4.12)$$

Corollary 3.4.10. *Let $(A, \succ, \prec)$ be a dendriform algebra. Then there exists a compatible quadri-algebra structure on $(A, \succ, \prec)$ such that $(A, \succ, \prec)$ is the associated horizontal dendriform algebra if and only if there exists an invertible $\mathcal{O}$-operator of $(A, \succ, \prec)$.*

Proof. If there exists an invertible $\mathcal{O}$-operator T of $(A, \succ, \prec)$ associated to a bimodule $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V)$, then by Proposition 3.4.6, there exists a quadri-algebra structure on V given by equation (3.4.8). Therefore we can define a quadri-algebra structure on A by equation (3.4.9) such that T is an isomorphism of quadri-algebras, that is,

$$x \searrow y = T(l_{\succ}(x)T^{-1}(y)),\ x \nearrow y = T(r_{\succ}(y)T^{-1}(x)),$$
$$x \swarrow y = T(l_{\prec}(x)T^{-1}(y)),\ x \nwarrow y = T(r_{\prec}(y)T^{-1}(x)),\quad \forall x, y \in A.$$

Moreover it is a compatible quadri-algebra structure on $(A, \succ, \prec)$ since for any $x, y \in A$, we have

$$x \succ y = T(T^{-1}(x) \succ T^{-1}(y)) = T(r_{\succ}(y)T^{-1}(x) + l_{\succ}(x)T^{-1}(y)) = x \nearrow y + x \searrow y,$$

$$x \prec y = T(T^{-1}(x) \prec T^{-1}(y)) = T(r_{\prec}(y)T^{-1}(x) + l_{\prec}(x)T^{-1}(y)) = x \nwarrow y + x \swarrow y.$$

Conversely, let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra and $(A, \succ, \prec)$ be the associated horizontal dendriform algebra. Then $(L_{\searrow}, R_{\nearrow}, L_{\swarrow}, R_{\nwarrow}, A)$ is a bimodule of $(A, \succ, \prec)$ and the identity map id is an $\mathcal{O}$-operator of $(A, \succ, \prec)$ associated to it. $\square$

Proposition 3.4.11. *Let $(A, \succ, \prec)$ be a dendriform algebra and $(A, *)$ be the associated associative algebra. If there is a nondegenerate symmetric 2-cocycle $\mathcal{B}$ of $(A, \succ, \prec)$, then there exists a compatible quadri-algebra structure on $(A, \succ, \prec)$ defined by (for any $x, y, z \in A$)*

$$\mathcal{B}(x \searrow y, z) = \mathcal{B}(y, z * x) = \mathcal{B}(y, z \succ x + z \prec x), \tag{3.4.13}$$

$$\mathcal{B}(x \nearrow y, z) = -\mathcal{B}(x, y \prec z), \tag{3.4.14}$$

$$\mathcal{B}(x \nwarrow y) = \mathcal{B}(x, y * z) = \mathcal{B}(x, y \succ z + y \prec z), \tag{3.4.15}$$

$$\mathcal{B}(x \swarrow y, z) = -\mathcal{B}(y, z \succ x). \tag{3.4.16}$$

such that $(A, \succ, \prec)$ is the associated horizontal dendriform algebra.

Proof. Since $\mathcal{B}$ is nondegenerate and symmetric, we define an invertible linear map $T : A \to A^*$ by

$$\langle x, T(y)\rangle = \langle T(x), y\rangle = \mathcal{B}(x, y), \ \ \forall x, y \in A.$$

Let $x, y, z \in A$. Since $\mathcal{B}$ is a 2-cocycle of $(A, \succ, \prec)$, we have

$$\begin{aligned} \langle T(x \succ y), z\rangle &= \mathcal{B}(x \succ y, z) = \mathcal{B}(y, z * x) - \mathcal{B}(x, y \succ z) \\ &= \langle T(y), z * x\rangle - \langle T(x), y \prec z\rangle = \langle R_*^*(x)T(y), z\rangle - \langle L_{\prec}^*(y)T(x), z\rangle. \end{aligned}$$

So

$$T(x \succ y) = R_*^*(x)T(y) - L_{\prec}^*(y)T(x), \ \ \forall x, y \in A.$$

Similarly, we show that

$$T(x \prec y) = L_*^*(y)T(x) - R_{\succ}^*(x)T(y), \ \ \forall x, y \in A.$$

Hence T^{-1} is an (invertible) $\mathcal{O}$-operator of the dendriform algebra $(A, \succ, \prec)$ associated to the bimodule $(R_*^*, -L_{\prec}^*, -R_{\succ}^*, L_*^*, A^*)$. Then by Corollary 3.4.10, there exists a compatible quadri-algebra structure on $(A, \succ, \prec)$ defined by

$$\begin{aligned} \mathcal{B}(x \searrow y, z) &= \langle T(x \searrow y), z\rangle = \langle T(T^{-1}(R_*^*(x)T(y))), z\rangle = \langle T(y), z * x\rangle = \mathcal{B}(y, z * x); \\ \mathcal{B}(x \nearrow y, z) &= \langle T(x \nearrow y), z\rangle = \langle T(T^{-1}(-L_{\prec}^*(y)T(x))), z\rangle = \langle T(x), -y \prec z\rangle = -\mathcal{B}(x, y \prec z); \\ \mathcal{B}(x \nwarrow y, z) &= \langle T(x \nwarrow y), z\rangle = \langle T(T^{-1}(L_*^*(y)T(x))), z\rangle = \langle T(x), y * z\rangle = \mathcal{B}(x, y * z); \\ \mathcal{B}(x \swarrow y, z) &= \langle T(x \swarrow y), z\rangle = \langle T(T^{-1}(-R_{\succ}^*(x)T(y))), z\rangle = \langle T(y), -z \succ y\rangle = -\mathcal{B}(y, z \succ x), \end{aligned}$$

such that $(A, \succ, \prec)$ is the associated horizontal dendriform algebra. $\square$

Proposition 3.4.12. (cf. Corollary 3.3.8) *Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra and $(A,\succ,\prec)$ be the associated horizontal dendriform algebra. Then r given by equation (3.3.4) is a symmetric solution of D-equation in the dendriform algebra $A\ltimes_{R^*_{\nearrow}+R^*_{\nwarrow},-L^*_{\swarrow},-R^*_{\nearrow},L^*_{\searrow}+L^*_{\swarrow}} A^*$, where $\{e_1,\cdots,e_n\}$ is a basis of A and $\{e_1^*,\cdots,e_n^*\}$ is its dual basis. Moreover there is a natural 2-cocycle $\mathcal{B}$ of the dendriform algebra $A\ltimes_{R^*_{\nearrow}+R^*_{\nwarrow},-L^*_{\swarrow},-R^*_{\nearrow},L^*_{\searrow}+L^*_{\swarrow}} A^*$ induced by $r^{-1}:A\oplus A^*\rightarrow (A\oplus A^*)^*$, which is given by equation (2.2.11).*

Proof. By Corollary 3.4.5, $(R^*_{\nearrow}+R^*_{\nwarrow},-L^*_{\swarrow},-R^*_{\nearrow},L^*_{\searrow}+L^*_{\swarrow},A^*)$ is the dual bimodule of the bimodule $(L_{\searrow},R_{\nearrow},L_{\swarrow},R_{\nwarrow},A)$ of the associated horizontal dendriform algebra $(A,\succ,\prec)$. Then the first half of the conclusion follows immediately from Theorem 3.3.5 and the fact that id is an $\mathcal{O}$-operator of $(A,\succ,\prec)$ associated to the bimodule $(L_{\searrow},R_{\nearrow},L_{\swarrow},R_{\nwarrow},A)$. The second half of the conclusion follows from Theorem 3.2.7. $\square$

At the end of this subsection, we give an algebraic equation on a quadri-algebra as follows.

Proposition 3.4.13. *Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra and $(A,\succ,\prec)$ be the associated horizontal dendriform algebra. Let $r\in A\otimes A$ be skew-symmetric. Then r is an $\mathcal{O}$-operator of $(A,\succ,\prec)$ associated to the bimodule $(R^*_{\nearrow}+R^*_{\nwarrow},-L^*_{\swarrow},-R^*_{\nearrow},L^*_{\searrow}+L^*_{\swarrow},A^*)$ if and only if r satisfies*

$$r_{13}\succ r_{23}=r_{23}\nearrow r_{12}+r_{23}\nwarrow r_{12}+r_{12}\swarrow r_{13},\tag{3.4.17}$$

$$r_{13}\prec r_{23}=-r_{23}\nearrow r_{12}-r_{12}\searrow r_{13}-r_{12}\swarrow r_{13}.\tag{3.4.18}$$

Proof. Let $\{e_1,\cdots,e_n\}$ be basis of A and $\{e_1^*,\cdots,e_n^*\}$ be its dual basis. Suppose that

$$e_i\searrow e_j=\sum_k a_{ij}^k e_k,\ e_i\nearrow e_j=\sum_k b_{ij}^k e_k,\ \ e_i\nwarrow e_j=\sum_k c_{ij}^k e_k,\ e_i\swarrow e_j=\sum_k d_{ij}^k e_k,$$

and $r=\sum\limits_{i,j} r_{ij}e_i\otimes e_j$, $r_{ij}=-r_{ji}$. Hence $r(e_i^*)=\sum\limits_k r_{ik}e_k$. Then r satisfies equation (3.4.17) if and only if (for any m,t,p)

$$\sum_{i,k}[r_{mi}r_{tk}(a_{ik}^p+b_{ik}^p)-r_{ip}r_{mk}(b_{ik}^t+c_{ik}^p)-r_{it}r_{kp}d_{ik}^m]=0.$$

The left-hand side of the above equation is precisely the coefficient of e_p in

$$r(e_m^*)\succ r(e_t^*)-r((R^*_{\nearrow}+R^*_{\nwarrow})(r(e_m^*))e_t^*-L^*_{\swarrow}(r(e_t^*))e_m^*).$$

On the other hand, r satisfies equation (3.4.18) if and only if (for any m,t,p)

$$\sum_{i,k}[r_{mi}r_{tk}(c_{ik}^p+d_{ik}^p)+r_{ip}r_{mk}b_{ik}^t+r_{it}r_{kp}(d_{ik}^m+a_{ik}^m)]=0.$$

The left-hand side of the above equation is precisely the coefficient of e_p in

$$r(e_m^*)\prec r(e_t^*)-r(-R^*_{\nearrow}(r(e_m^*))e_t^*+(L^*_{\searrow}+L^*_{\swarrow})(r(e_t^*))e_m^*).$$

Therefore the conclusion follows. $\square$

Corollary 3.4.14. *Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra and $(A,\succ,\prec)$ be the associated horizontal dendriform algebra. Let $r\in A\otimes A$. If r satisfies equations (3.4.17) and (3.4.18), then r satisfies*

$$r_{13}*r_{23}=r_{23}\nwarrow r_{12}-r_{12}\searrow r_{13}. \tag{3.4.19}$$

Proof. In fact, equation (3.4.19) is the sum of equations (3.4.17) and (3.4.18). □

4. $\mathcal{O}$-operators of quadri-algebras

4.1. **Bimodule of quadri-algebras.**

Definition 4.1.1. Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra and V be a vector space. Let $l_{\searrow},r_{\searrow},l_{\nearrow},r_{\nearrow},l_{\nwarrow},r_{\nwarrow},l_{\swarrow},r_{\swarrow}:A\to gl(V)$ be eight linear maps. V (or $(l_{\searrow},r_{\searrow},l_{\nearrow},r_{\nearrow},l_{\nwarrow},r_{\nwarrow},l_{\swarrow},r_{\swarrow})$, or $(l_{\searrow},r_{\searrow},l_{\nearrow},r_{\nearrow},l_{\nwarrow},r_{\nwarrow},l_{\swarrow},r_{\swarrow},V)$) is called a *bimodule* of A if the following equations hold (for any $x,y\in A$):

$$l_{\nwarrow}(x\nwarrow y)=l_{\nwarrow}(x)l_{*}(y);\quad r_{\nwarrow}(y)l_{\nwarrow}(x)=l_{\nwarrow}(x)r_{*}(y);\quad r_{\nwarrow}(y)r_{\nwarrow}(x)=r_{\nwarrow}(x*y); \tag{4.1.1}$$

$$l_{\nwarrow}(x\swarrow y)=l_{\swarrow}(x)l_{\wedge}(y);\quad r_{\nwarrow}(y)l_{\swarrow}(x)=l_{\swarrow}(x)r_{\wedge}(y);\quad r_{\nwarrow}(y)r_{\swarrow}(x)=r_{\swarrow}(x\wedge y); \tag{4.1.2}$$

$$l_{\swarrow}(x\prec y)=l_{\swarrow}(x)l_{\vee}(y);\quad r_{\swarrow}(y)l_{\prec}(x)=l_{\swarrow}(x)r_{\vee}(y);\quad r_{\swarrow}(y)r_{\prec}(x)=r_{\swarrow}(x\vee y); \tag{4.1.3}$$

$$l_{\nwarrow}(x\nearrow y)=l_{\nearrow}(x)l_{\prec}(y);\quad r_{\nwarrow}(y)l_{\nearrow}(x)=l_{\nearrow}(x)r_{\prec}(y);\quad r_{\nwarrow}(y)r_{\nearrow}(x)=r_{\nearrow}(x\prec y); \tag{4.1.4}$$

$$l_{\nwarrow}(x\searrow y)=l_{\searrow}(x)l_{\nwarrow}(y);\quad r_{\nwarrow}(y)l_{\searrow}(x)=l_{\searrow}(x)r_{\nwarrow}(y);\quad r_{\nwarrow}(y)r_{\searrow}(x)=r_{\searrow}(x\nwarrow y); \tag{4.1.5}$$

$$l_{\swarrow}(x\succ y)=l_{\searrow}(x)l_{\swarrow}(y);\quad r_{\swarrow}(y)l_{\succ}(x)=l_{\searrow}(x)r_{\swarrow}(y);\quad r_{\swarrow}(y)r_{\succ}(x)=r_{\searrow}(x\swarrow y); \tag{4.1.6}$$

$$l_{\nearrow}(x\wedge y)=l_{\nearrow}(x)l_{\succ}(y);\quad r_{\nearrow}(y)l_{\wedge}(x)=l_{\nearrow}(x)r_{\succ}(y);\quad r_{\nearrow}(y)r_{\wedge}(x)=r_{\nearrow}(x\succ y); \tag{4.1.7}$$

$$l_{\nearrow}(x\vee y)=l_{\searrow}(x)l_{\nearrow}(y);\quad r_{\nearrow}(y)l_{\vee}(x)=l_{\searrow}(x)r_{\nearrow}(y);\quad r_{\nearrow}(y)r_{\vee}(x)=r_{\searrow}(x\nearrow y); \tag{4.1.8}$$

$$l_{\searrow}(x*y)=l_{\searrow}(x)l_{\searrow}(y);\quad r_{\searrow}(y)l_{*}(x)=l_{\searrow}(x)r_{\searrow}(y);\quad r_{\searrow}(y)r_{*}(x)=r_{\searrow}(x\searrow y), \tag{4.1.9}$$

where

$$x\succ y=x\nearrow y+x\searrow y,\ x\prec y=x\nwarrow y+x\swarrow y, \tag{4.1.10}$$

$$l_{\succ}=l_{\nearrow}+l_{\searrow},\ r_{\succ}=r_{\nearrow}+r_{\searrow},\ l_{\prec}=l_{\nwarrow}+l_{\swarrow},\ r_{\prec}=r_{\nwarrow}+r_{\swarrow}, \tag{4.1.11}$$

$$x\vee y=x\searrow y+x\swarrow y,\ x\wedge y=x\nearrow y+x\nwarrow y, \tag{4.1.12}$$

$$l_{\vee}=l_{\searrow}+l_{\swarrow},\ r_{\vee}=r_{\searrow}+r_{\swarrow},\ l_{\wedge}=l_{\nearrow}+l_{\nwarrow},\ r_{\wedge}=r_{\nearrow}+r_{\nwarrow}, \tag{4.1.13}$$

$$x*y=x\searrow y+x\nearrow y+x\nwarrow y+x\swarrow y, \tag{4.1.14}$$

$$l_{*}=l_{\searrow}+l_{\nearrow}+l_{\nwarrow}+l_{\swarrow},\ r_{*}=r_{\searrow}+r_{\nearrow}+r_{\nwarrow}+r_{\swarrow}. \tag{4.1.15}$$

According to [Sc], $(l_{\searrow},r_{\searrow},l_{\nearrow},r_{\nearrow},l_{\nwarrow},r_{\nwarrow},l_{\swarrow},r_{\swarrow},V)$ is a bimodule of a quadri-algebra $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ if and only if there exists a quadri-algebra structure on the direct sum $A\oplus V$ of the underlying vector spaces of A and V given by $(\forall x,y\in A,u,v\in V)$

$$\begin{aligned}&(x+u)\searrow(y+v)=x\searrow y+l_{\searrow}(x)v+r_{\searrow}(y)u,\quad (x+u)\nearrow(y+v)=x\nearrow y+l_{\nearrow}(x)v+r_{\nearrow}(y)u,\\ &(x+u)\nwarrow(y+v)=x\nwarrow y+l_{\nwarrow}(x)v+r_{\nwarrow}(y)u,\quad (x+u)\swarrow(y+v)=x\swarrow y+l_{\swarrow}(x)v+r_{\swarrow}(y)u.\end{aligned} \tag{4.1.16}$$

We denote it by $A\ltimes_{l_{\searrow},r_{\searrow},l_{\nearrow},r_{\nearrow},l_{\nwarrow},r_{\nwarrow},l_{\swarrow},r_{\swarrow}}V$.

Proposition 4.1.2. *Let $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ be a bimodule of a quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. Let $(A, \succ, \prec)$ be the associated horizontal dendriform algebra.*

(1) $(l_{\searrow}, r_{\nearrow}, l_{\swarrow}, r_{\nwarrow}, V)$ is a bimodule of $(A, \succ, \prec)$.

(2) $(l_{\nearrow} + l_{\searrow}, r_{\nearrow} + r_{\searrow}, l_{\nwarrow} + l_{\swarrow}, r_{\nwarrow} + r_{\swarrow}, V)$ is a bimodule of $(A, \succ, \prec)$.

(3) For any bimodule $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V)$ of $(A, \succ, \prec)$, $(l_{\succ}, 0, 0, r_{\succ}, 0, r_{\prec}, l_{\prec}, 0, V)$ is a bimodule of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

(4) Both $(l_{\searrow}, 0, 0, r_{\nearrow}, 0, r_{\nwarrow}, l_{\swarrow}, 0, V)$ and $(l_{\nearrow} + l_{\searrow}, 0, 0, r_{\nearrow} + r_{\searrow}, 0, r_{\nwarrow} + r_{\swarrow}, l_{\nwarrow} + l_{\swarrow}, 0, V)$ are bimodules of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

(5) The quadri-algebras $A \ltimes_{l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}} V$ and $A \ltimes_{l_{\nearrow}+l_{\searrow}, 0, 0, r_{\nearrow}+r_{\searrow}, 0, r_{\nwarrow}+r_{\swarrow}, l_{\nwarrow}+l_{\swarrow}, 0} V$ have the same associated horizontal dendriform algebra $A \ltimes_{l_{\nearrow}+l_{\searrow}, r_{\nearrow}+r_{\searrow}, l_{\nwarrow}+l_{\swarrow}, r_{\nwarrow}+r_{\swarrow}} V$.

Proof. (1) follows from the following correspondences of equations:

(3.1.1) $\Longleftrightarrow$ (4.1.3-1) (3.1.2) $\Longleftrightarrow$ (4.1.2-2); (3.1.3) $\Longleftrightarrow$ (4.1.1-3);
(3.1.4) $\Longleftrightarrow$ (4.1.6-1); (3.1.5) $\Longleftrightarrow$ (4.1.5-2); (3.1.6) $\Longleftrightarrow$ (4.1.4-3);
(3.1.7) $\Longleftrightarrow$ (4.1.9-1); (3.1.8) $\Longleftrightarrow$ (4.1.8-2); (3.1.9) $\Longleftrightarrow$ (4.1.7-3).

(2) follows from the following correspondences of equations:

(3.1.1) $\Longleftrightarrow$ (4.1.1-1)+(4.1.2-1)+(4.1.3-1); (3.1.2) $\Longleftrightarrow$ (4.1.1-2)+(4.1.2-2)+(4.1.3-2);
(3.1.3) $\Longleftrightarrow$ (4.1.1-3)+(4.1.2-3)+(4.1.3-3); (3.1.4) $\Longleftrightarrow$ (4.1.4-1)+(4.1.5-1)+(4.1.6-1);
(3.1.5) $\Longleftrightarrow$ (4.1.4-2)+(4.1.5-2)+(4.1.6-2); (3.1.6) $\Longleftrightarrow$ (4.1.4-3)+(4.1.5-3)+(4.1.6-3);
(3.1.7) $\Longleftrightarrow$ (4.1.7-1)+(4.1.8-1)+(4.1.9-1); (3.1.8) $\Longleftrightarrow$ (4.1.7-2)+(4.1.8-2)+(4.1.9-2);
(3.1.9) $\Longleftrightarrow$ (4.1.7-3)+(4.1.8-3)+(4.1.9-3).

(3) In this case $r_{\searrow} = l_{\nearrow} = l_{\nwarrow} = r_{\swarrow} = 0, l_{\searrow} = l_{\succ}, r_{\nearrow} = r_{\succ}, r_{\nwarrow} = r_{\prec}, l_{\swarrow} = l_{\prec}$, it is obvious that the fact that $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec})$ is a bimodule of $(A, \succ, \prec)$ corresponds to the equations appearing in (1) and the other equations hold (both sides are zero). So (3) holds.

(4) follows immediately from (1), (2) and (3).

(5) follows immediately from (4). □

By the above conclusion and Proposition 3.1.2, we have the following results.

Corollary 4.1.3. *Let $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ be a bimodule of a quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. Let $(A, *)$ be the associated associative algebra.*

(1) $(l_{\searrow}, r_{\nwarrow}, V)$,$(l_{\nearrow} + l_{\searrow}, r_{\nwarrow} + r_{\swarrow}, V)$,$(l_{\searrow} + l_{\swarrow}, r_{\nearrow} + r_{\nwarrow}, V)$,$(l_{\nearrow} + l_{\searrow} + l_{\nwarrow} + l_{\swarrow}, r_{\nearrow} + r_{\searrow} + r_{\nwarrow} + r_{\swarrow}, V)$ are bimodules of $(A, *)$.

(2) For any bimodule (l, r, V) of $(A, *)$, $(l, 0, 0, 0, 0, r, 0, 0)$ is a bimodule of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$.

Proposition 4.1.4. *Let $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$ be a bimodule of a quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. Then $(r^*_{*}, l^*_{\nwarrow}, -r^*_{\vee}, -l^*_{\prec}, r^*_{\searrow}, l^*_{*}, -r^*_{\succ}, -l^*_{\wedge}, A^*)$ is a bimodule of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. We call it the dual bimodule of $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$.*

Proof. This conclusion can be obtained by a direct checking on equations (4.1.1)-(4.1.9). We give another approach by using the relations between bimodules of a quadri-algebra $(A, \searrow, \nearrow, \nwarrow,$

$\swarrow$) and the associated horizontal dendriform algebra $(A,\succ,\prec)$ with the (known) dual bimodules of $(A,\succ,\prec)$. Let $(\bar{l}_\searrow,\bar{r}_\searrow,\bar{l}_\nearrow,\bar{r}_\nearrow,\bar{l}_\nwarrow,\bar{r}_\nwarrow,\bar{l}_\swarrow,\bar{r}_\swarrow,V^*)$ be the dual bimodule of $(l_\searrow,r_\searrow,l_\nearrow,r_\nearrow,$ $l_\nwarrow,r_\nwarrow,l_\swarrow,r_\swarrow,V)$. Then by Proposition 4.1.2, both

$$(\bar{l}_\searrow,\bar{r}_\nearrow,\bar{l}_\swarrow,\bar{r}_\nwarrow,V^*)\ \text{ and }\ (\bar{l}_\nearrow+\bar{l}_\searrow,\bar{r}_\nearrow+\bar{r}_\searrow,\bar{l}_\nwarrow+\bar{l}_\swarrow,\bar{r}_\nwarrow+\bar{r}_\swarrow,V^*)$$

are bimodules of $(A,\succ,\prec)$. On the other hand, since both $(l_\searrow,r_\nearrow,l_\swarrow,r_\nwarrow,V)$ and $(l_\nearrow+l_\searrow,r_\nearrow+r_\searrow,l_\nwarrow+l_\swarrow,r_\nwarrow+r_\swarrow,V)$ are bimodules of $(A,\succ,\prec)$, their dual bimodules are

$$(r^*_\wedge,-l^*_\swarrow,-r^*_\nearrow,l^*_\vee,V^*),\ \text{ and }\ (r^*_*,-l^*_\prec,-r^*_\succ,l^*_*,V^*)$$

respectively. Hence we have the following equations

$$\bar{l}_\searrow=r^*_*,\bar{r}_\nearrow=-l^*_\prec,\bar{l}_\swarrow=-r^*_\succ,\bar{r}^*_\nwarrow=l^*_*;$$

$$\bar{l}_\nearrow+\bar{l}_\searrow=r^*_\wedge,\bar{r}_\nearrow+\bar{r}_\searrow=-l^*_\swarrow,\bar{l}_\nwarrow+\bar{l}_\swarrow=-r^*_\nearrow,\bar{r}_\nwarrow+\bar{r}_\swarrow=l^*_\vee.$$

Therefore $(r^*_*,l^*_\nwarrow,-r^*_\vee,-l^*_\prec,r^*_\searrow,l^*_*,-r^*_\succ,-l^*_\wedge,A^*)$ is a bimodule of $(A,\searrow,\nearrow,\nwarrow,\swarrow)$. □

By Propositions 4.1.2 and 4.1.4, the following conclusion is obvious.

Corollary 4.1.5. *Let $(l_\searrow,r_\searrow,l_\nearrow,r_\nearrow,l_\nwarrow,r_\nwarrow,l_\swarrow,r_\swarrow,V)$ be a bimodule of a quadri-algebra $(A,$ $\searrow,\nearrow,\nwarrow,\swarrow)$.*

(1) Both $(r^*_*,0,0-l^*_\prec,0,l^*_*,-r^*_\succ,0,V^*)$ and $(r^*_\wedge,0,0,-l^*_\swarrow,0,l^*_\vee,-r^*_\nearrow,0,V^*)$ are bimodules of $(A,\searrow,\nearrow,\nwarrow,\swarrow)$.

(2) Both $(r^*_*,-l^*_\prec,-r^*_\succ,l^*_*,V^*)$ and $(r^*_\wedge,-l^*_\swarrow,-r^*_\nearrow,l^*_\vee,V^*)$ are bimodules of the associated horizontal dendriform algebra $(A,\succ,\prec)$.

Example 4.1.6. Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra. Then

$$(L_\searrow,R_\searrow,L_\nearrow,R_\nearrow,L_\nwarrow,R_\nwarrow,L_\swarrow,R_\swarrow,A),\ (L_\searrow,0,0,R_\nearrow,0,R_\nwarrow,L_\swarrow,0,A),$$

$$(L_\nearrow+L_\searrow,0,0,R_\nearrow+R_\searrow,0,R_\nwarrow+R_\swarrow,L_\nwarrow+L_\swarrow,0,A)$$

are bimodules of $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ and the first one is called the regular bimodule of $(A,\searrow,\nearrow,$ $\nwarrow,\swarrow)$. On the other hand,

$$(R^*_*,L^*_\nwarrow,-R^*_\vee,-L^*_\prec,R^*_\searrow,L^*_*,-R^*_\succ,-L^*_\wedge,A^*),\ (R^*_*,0,0-L^*_\prec,0,L^*_*,-R^*_\succ,0,A^*),$$

$$(R^*_\wedge,0,0,-L^*_\swarrow,0,L^*_\vee,-R^*_\nearrow,0,A^*)$$

are bimodules of $(A,\searrow,\nearrow,\nwarrow,\swarrow)$, too.

4.2. **$\mathcal{O}$-operators of quadri-algebras and Q-equation.**

Definition 4.2.1. Let $(l_\searrow,r_\searrow,l_\nearrow,r_\nearrow,l_\nwarrow,r_\nwarrow,l_\swarrow,r_\swarrow,V)$ be a bimodule of a quadri-algebra $(A,\searrow,\nearrow,\nwarrow,\swarrow)$. A linear map $T:V\to A$ is called an *$\mathcal{O}$-operator of* $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ *associated to* $(l_\searrow,r_\searrow,l_\nearrow,r_\nearrow,l_\nwarrow,r_\nwarrow,l_\swarrow,r_\swarrow,V)$ if T satisfies

$$T(u)\searrow T(v)=T(l_\searrow(T(u))v+r_\searrow(T(v)u)),\tag{4.2.1}$$

$$T(u)\nearrow T(v)=T(l_\nearrow(T(u))v+r_\nearrow(T(v)u)),\tag{4.2.2}$$

$$T(u) \nwarrow T(v) = T(l_{\nwarrow}(T(u))v + r_{\nwarrow}(T(v)u)), \tag{4.2.3}$$
$$T(u) \swarrow T(v) = T(l_{\swarrow}(T(u))v + r_{\swarrow}(T(v)u)), \tag{4.2.4}$$

for any $u, v \in V$.

The following result is obvious.

Corollary 4.2.2. *Let T be an $\mathcal{O}$-operator of a quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to a bimodule $(l_{\searrow}, r_{\searrow}, l_{\nearrow}, r_{\nearrow}, l_{\nwarrow}, r_{\nwarrow}, l_{\swarrow}, r_{\swarrow}, V)$.*

(1) T is an $\mathcal{O}$-operator of the associated horizontal dendriform algebra $(A, \succ, \prec)$ associated to $(l_{\nearrow} + l_{\searrow}, r_{\nearrow} + r_{\searrow}, l_{\nwarrow} + l_{\swarrow}, r_{\nwarrow} + r_{\swarrow}, V)$.

(2) T is an $\mathcal{O}$-operator of the associated associative algebra $(A, *)$ associated to $(l_{\nearrow} + l_{\searrow} + l_{\nwarrow} + l_{\swarrow}, r_{\nearrow} + r_{\searrow} + r_{\nwarrow} + r_{\swarrow}, V)$.

Proposition 4.2.3. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. Let $r \in A \otimes A$ be skew-symmetric. Then r is an $\mathcal{O}$-operator of the quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to $(R^*_*, L^*_{\nwarrow}, -R^*_{\vee}, -L^*_{\prec}, R^*_{\searrow}, L^*_*, -R^*_{\succ}, -L^*_{\wedge}, A^*)$ if and only if r satisfies*

$$r_{13} \searrow r_{23} = r_{23} * r_{12} - r_{12} \nwarrow r_{13}, \tag{4.2.5}$$
$$r_{13} \nearrow r_{23} = -r_{23} \vee r_{12} + r_{12} \prec r_{13}, \tag{4.2.6}$$
$$r_{13} \nwarrow r_{23} = r_{23} \searrow r_{12} - r_{12} * r_{13}, \tag{4.2.7}$$
$$r_{13} \swarrow r_{23} = -r_{23} \succ r_{12} + r_{12} \wedge r_{13}. \tag{4.2.8}$$

Proof. The conclusion follows from a similar proof as of Proposition 3.4.13. □

Lemma 4.2.4. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra and $r \in A \otimes A$. Let r be skew-symmetric.*

(1) Equation (3.4.17) holds if and only if equation (4.2.8) holds.

(2) Equation (3.4.18) holds if and only if equation (4.2.6) holds.

(3) Equation (3.4.19) holds if and only if equation (4.2.5) holds.

(4) equation (3.4.19) holds if and only if equation (4.2.7) holds.

Proof. Let $\sigma_{123}, \sigma_{132} : A \otimes A \otimes A \to A \otimes A \otimes A$ be two linear maps given by

$$\sigma_{123}(x \otimes y \otimes z) = z \otimes x \otimes y, \quad \sigma_{132}(x \otimes y \otimes z) = y \otimes z \otimes x, \quad \forall\, x, y, z \in A,$$

respectively. Then we have the following equations.

$$\begin{aligned}
\sigma_{132}(r_{13} \succ r_{23} - r_{23} \wedge r_{12} - r_{12} \swarrow r_{13}) &= r_{32} \succ r_{12} - r_{12} \wedge r_{31} - r_{31} \swarrow r_{32} \\
&= -r_{23} \succ r_{12} + r_{12} \wedge r_{13} - r_{13} \swarrow r_{23}; \\
\sigma_{123}(r_{13} \prec r_{23} + r_{23} \nearrow r_{12} + r_{12} \vee r_{23}) &= r_{21} \prec r_{31} + r_{31} \nearrow r_{23} + r_{23} \vee r_{21} \\
&= r_{12} \prec r_{13} - r_{13} \nearrow r_{23} - r_{23} \vee r_{12}; \\
\sigma_{132}(r_{13} * r_{23} - r_{23} \nwarrow r_{12} + r_{12} \searrow r_{13}) &= r_{32} * r_{12} - r_{12} \nwarrow r_{31} + r_{31} \searrow r_{32} \\
&= -r_{23} * r_{12} + r_{12} \nwarrow r_{13} + r_{13} \searrow r_{23};
\end{aligned}$$

$$\begin{aligned}\sigma_{123}(r_{13}*r_{23}-r_{23}\nwarrow r_{12}+r_{12}\searrow r_{13}) &= r_{21}*r_{31}-r_{31}\nwarrow r_{23}+r_{23}\searrow r_{21}\\ &= r_{12}*r_{13}+r_{13}\nwarrow r_{23}-r_{23}\searrow r_{12}.\end{aligned}$$

Therefore the conclusion follows immediately. □

By Propositions 3.4.13 and 4.2.3 and Lemma 4.2.4, we have the following conclusion.

Corollary 4.2.5. *Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra and $r\in A\otimes A$. Let r be skew-symmetric. Then the following conditions are equivalent.*

(1) r is an $\mathcal{O}$-operator of the associated horizontal dendriform algebra $(A,\succ,\prec)$ associated to the bimodule $(R^*_{\nearrow}+R^*_{\nwarrow},-L^*_{\swarrow},-R^*_{\nearrow},L^*_{\searrow}+L^*_{\swarrow},A^*)$.

(2) r is an $\mathcal{O}$-operator of the quadri-algebra $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ associated to the bimodule $(R^*_{*},L^*_{\nwarrow},-R^*_{\vee},-L^*_{\prec},R^*_{\searrow},L^*_{*},-R^*_{\succ},-L^*_{\wedge},A^*)$.

(3) r satisfies equations (3.4.17) and (3.4.18) in $(A,\searrow,\nearrow,\nwarrow,\swarrow)$.

(4) r satisfies equations (4.2.6) and (4.2.8) in $(A,\searrow,\nearrow,\nwarrow,\swarrow)$.

Definition 4.2.6. Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra and $r\in A\otimes A$. A set of equations (3.4.17) and (3.4.18) is called *Q-equation in $(A,\searrow,\nearrow,\nwarrow,\swarrow)$*.

Remark 4.2.7. Due to Corollary 4.2.5, it is reasonable to regard the Q-equation (a set of equations) in a quadri-algebra as an analogue of the classical Yang-Baxter equation in a Lie algebra.

With a similar discussion as in subsection 3.3, we have the following results (see Proposition 3.3.5, Corollaries 3.3.6 and 3.3.8).

Theorem 4.2.8. *Let $(l_{\searrow},r_{\searrow},l_{\nearrow},r_{\nearrow},l_{\nwarrow},r_{\nwarrow},l_{\swarrow},r_{\swarrow},V)$ be a bimodule of a quadri-algebra $(A,\searrow,\nearrow,\nwarrow,\swarrow)$. Let $(r^*_{*},l^*_{\nwarrow},-r^*_{\vee},-l^*_{\prec},r^*_{\searrow},l^*_{*},-r^*_{\succ},-l^*_{\wedge},A^*)$ be the dual bimodule given by Proposition 4.1.4. Let $T:V\to A$ be a linear map identified as an element in $A\otimes V^*$ which is in the underlying vector space of*

$$(A\ltimes_{r^*_{*},l^*_{\nwarrow},-r^*_{\vee},-l^*_{\prec},r^*_{\searrow},l^*_{*},-r^*_{\succ},-l^*_{\wedge}}V^*)\otimes(A\ltimes_{r^*_{*},l^*_{\nwarrow},-r^*_{\vee},-l^*_{\prec},r^*_{\searrow},l^*_{*},-r^*_{\succ},-l^*_{\wedge}}V^*).$$

Then $r=T-\sigma(T)$ is a skew-symmetric solution of Q-equation in the quadri-algebra $A\ltimes_{r^*_{*},l^*_{\nwarrow},-r^*_{\vee},-l^*_{\prec},r^*_{\searrow},l^*_{*},-r^*_{\succ},-l^*_{\wedge}}V^*$ if and only if T is an $\mathcal{O}$-operator of the quadri-algebra $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ associated to $(l_{\searrow},r_{\searrow},l_{\nearrow},r_{\nearrow},l_{\nwarrow},r_{\nwarrow},l_{\swarrow},r_{\swarrow},V)$.

Corollary 4.2.9. *Let $(A,\succ,\prec)$ be a dendriform algebra. Let $(l_{\succ},r_{\succ},l_{\prec},r_{\prec},V)$ be a bimodule of A and $(r^*_{\succ}+r^*_{\prec},-l^*_{\prec},-r^*_{\succ},l^*_{\succ}+l^*_{\prec},V^*)$ be the dual bimodule given in Proposition 3.1.3. Suppose that $T:V\to A$ is an $\mathcal{O}$-operator of $(A,\succ,\prec)$ associated to $(l_{\succ},r_{\succ},l_{\prec},r_{\prec},V)$. Then $r=T-\sigma(T)$ is a skew-symmetric solution of Q-equation in the quadri-algebra*

$$T(V)\ltimes_{r^*_{\succ}+r^*_{\prec},0,0,-l^*_{\prec},0,l^*_{\succ}+l^*_{\prec},-r^*_{\succ},0}V^*,$$

where $T(V)\subset A$ is a quadri-algebra given by equation (3.4.9) and $(r_{\succ}^*+r_{\prec}^*,0,0,-l_{\prec}^*,0,l_{\succ}^*+l_{\prec}^*,-r_{\succ}^*,0,V^*)$ is its bimodule since its associated horizontal dendriform algebra $T(V)$ is a dendriform subalgebra of $(A,\succ,\prec)$, and T can be identified as an element in $T(V)\otimes V^*$ which is in the underlying vector space of

$$(T(V)\ltimes_{r_{\succ}^*+r_{\prec}^*,0,0,-l_{\prec}^*,0,l_{\succ}^*+l_{\prec}^*,-r_{\succ}^*,0,}V^*)\otimes(T(V)\ltimes_{r_{\succ}^*+r_{\prec}^*,0,0,-l_{\prec}^*,0,l_{\succ}^*+l_{\prec}^*,-r_{\succ}^*,0,}V^*).$$

Corollary 4.2.10. (cf. Corollary 4.4.13) *Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra. Then r given by equation (2.2.9) is a skew-symmetric solution of Q-equation in the quadri-algebra $A\ltimes_{R_{\wedge}^*,0,0,-L_{\swarrow}^*,0,L_{\vee}^*,-R_{\nearrow}^*,0}A^*$, where $\{e_1,\cdots,e_n\}$ is a basis of A and $\{e_1^*,\cdots,e_n^*\}$ is its dual basis.*

4.3. **Bilinear forms on quadri-algebras and Q-equation.** In this subsection, we consider the (symmetric and skew-symmetric) bilinear forms on quadri-algebras as we have done in subsections 2.2 and 3.2.

Lemma 4.3.1. *Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra. Let $\mathcal{B}$ be a symmetric bilinear form on A. Set*

$$\mathcal{B}(x,y\succ z)=\mathcal{B}(x\wedge y,z),\tag{4.3.1}$$

$$\mathcal{B}(y,z\prec x)=\mathcal{B}(x\vee y,z),\tag{4.3.2}$$

$$\mathcal{B}(x\searrow y,z)=\mathcal{B}(x,y\nwarrow z),\tag{4.3.3}$$

$$\mathcal{B}(x\nearrow y,z)+\mathcal{B}(z\swarrow x,y)+\mathcal{B}(y*z,x)=0,\tag{4.3.4}$$

$$\mathcal{B}(x\searrow y,z)+\mathcal{B}(y\nearrow z,x)-\mathcal{B}(z\succ x,y)=0,\tag{4.3.5}$$

$$\mathcal{B}(x\nwarrow y,z)+\mathcal{B}(z\swarrow x,y)-\mathcal{B}(y\prec z,x)=0,\tag{4.3.6}$$

for any $x,y,z\in A$. Then any two equations of the three equations in the following sets can imply the third equation.

(1) *Equations (3.4.14), (3.4.15) and (4.3.1);*
(2) *Equations (3.4.13), (3.4.16) and (4.3.2);*
(3) *Equations (3.4.13), (3.4.15) and (4.3.3);*
(4) *Equations (3.4.14), (3.4.16) and (4.3.4);*
(5) *Equations (3.4.13), (3.4.14) and (4.3.5);*
(6) *Equations (3.4.15), (3.4.16) and (4.3.6).*

Proof. It is straightforward. □

Definition 4.3.2. Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra. A symmetric bilinear form $\mathcal{B}$ on A is called *invariant* if $\mathcal{B}$ satisfies equations (3.4.13)-(3.4.16).

Corollary 4.3.3. *Any symmetric invariant bilinear form on a quadri-algebra is a 2-cocycle of the associated horizontal dendriform algebra.*

Proof. It follows immediately from the sum of equations (3.4.13)-(3.4.16). □

By Proposition 3.4.11 and Corollary 4.3.3, we have the following result.

Corollary 4.3.4. *Let $(A,\succ,\prec)$ be a dendriform algebra and $\mathcal{B}$ be a nondegenerate symmetric bilinear form. Then $\mathcal{B}$ is a 2-cocycle of $(A,\succ,\prec)$ if and only if $\mathcal{B}$ is invariant on the compatible quadri-algebra structure given by equations (3.4.13)-(3.4.16) which $(A,\succ,\prec)$ is the associated horizontal dendriform algebra.*

Remark 4.3.5. In Remark 4.2.7, we have seen that the Q-equation in a quadri-algebra is an analogue of the classical Yang-Baxter equation as an $\mathcal{O}$-operator of the quadri-algebra (or equivalently, its associated horizontal dendriform algebra) associated to certain dual bimodule. On the other hand, similar to the associative Yang-Baxter equation in an associative algebra coming from the double construction of the nondegenerate symmetric invariant bilinear forms on associative algebras and D-equation in a dendriform algebra coming from the double constructions of the nondegenerate (skew-symmetric) invariant bilinear forms on dendriform algebras (or equivalently, the double construction of the nondegenerate Connes cocycles on associative algebras), it is quite reasonable to believe that the Q-equation in a quadri-algebra should be related to certain "double construction" of the nondegenerate (symmetric) invariant bilinear forms on quadri-algebras (or equivalently, the double construction of the nondegenerate 2-cocycles on dendriform algebras). In fact, such a conclusion has been proved in [Ni].

Next we turn to the study of skew-symmetric bilinear forms on quadri-algebras.

Theorem 4.3.6. *Let $(A,\searrow,\nearrow,\nwarrow,\swarrow)$ be a quadri-algebra and $r\in A\otimes A$. Suppose that r is skew-symmetric and nondegenerate. Then r is a solution of Q-equation in A if and only if the inverse of the isomorphism $A^*\to A$ induced by r, regarded as a bilinear form ω on A (that is, $\omega(x,y)=\langle r^{-1}x,y\rangle$ for any $x,y\in A$) satisfies (for any $x,y,z\in A$)*

$$\omega(z,x\succ y)-\omega(x,y\swarrow z)+\omega(y,z\wedge x)=0,\ \omega(z,x\prec y)+\omega(x,y\vee z)-\omega(y,z\nearrow x)=0. \quad (4.3.7)$$

Proof. Let $r=\sum\limits_i a_i\otimes b_i$. Since r is skew-symmetric, we have $\sum\limits_i a_i\otimes b_i=-\sum\limits_i b_i\otimes a_i$. Therefore $r(v^*)=-\sum\limits_i\langle v^*,a_i\rangle b_i=\sum\limits_i\langle v^*,b_i\rangle a_i$ for any $v^*\in A^*$. Since r is nondegenerate, for any $x,y,z\in A$, there exist $u^*,v^*,w^*\in A^*$ such that $x=r(u^*),y=r(v^*),z=r(w^*)$. Therefore

$$\begin{aligned}
\langle u^*\otimes v^*\otimes w^*,r_{13}\succ r_{23}\rangle &= \sum_{i,j}\langle u^*,a_i\rangle\langle v^*,a_j\rangle\langle w^*,b_i\succ b_j\rangle\\
&= \langle r(u^*)\succ r(v^*),w^*\rangle=\omega(z,x\succ y);\\
\langle u^*\otimes v^*\otimes w^*,r_{23}\wedge r_{12}\rangle &= \sum_{i,j}\langle u^*,a_j\rangle\langle v^*,a_i\wedge b_j\rangle\langle w^*,b_i\succ b_j\rangle\\
&= -\langle r(w^*)\wedge r(u^*),v^*\rangle=-\omega(y,z\wedge x);\\
\langle u^*\otimes v^*\otimes w^*,r_{12}\swarrow r_{23}\rangle &= \sum_{i,j}\langle u^*,a_i\swarrow a_j\rangle\langle v^*,b_i\rangle\langle w^*,b_j\succ b_j\rangle\\
&= \langle r(v^*)\swarrow r(w^*),u^*\rangle=\omega(x,y\swarrow z).
\end{aligned}$$

So r satisfies equation (3.4.17) if and only if ω satisfies

$$\omega(z,x\succ y)-\omega(x,y\swarrow z)+\omega(y,z\wedge x)=0,\quad \forall x,y,z\in A.$$

Similarly, r satisfies equation (3.4.18) if and only if ω satisfies

$$\omega(z, x \prec y) + \omega(x, y \vee z) - \omega(y, z \nearrow x) = 0,\ \forall x, y, z \in A.$$

$\square$

Definition 4.3.7. Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. A skew-symmetric bilinear form ω on A is called a *2-cocycle* if ω satisfies equation (4.3.7).

By Corollary 4.2.10 and Theorem 4.3.6, we have the following conclusion.

Corollary 4.3.8. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. Then the skew-symmetric solution of Q-equation in the quadri-algebra $A \ltimes_{R^*_{\wedge},0,0,-L^*_{\swarrow},0,L^*_{\vee},-R^*_{\nearrow},0} A^*$ given by equation (2.2.9) induces a natural 2-cocycle ω on $A \ltimes_{R^*_{\wedge},0,0,-L^*_{\swarrow},0,L^*_{\vee},-R^*_{\nearrow},0} A^*$ by $r^{-1} : A \oplus A^* \to (A \oplus A^*)^*$, which is given by equation (2.2.10).*

4.4. $\mathcal{O}$-operators of quadri-algebras and octo-algebras.

Definition 4.4.1. ([Le3]) Let A be a vector space with eight bilinear products denoted by

$$\searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2 : A \otimes A \to A.$$

$(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is called an *octo-algebra* if for any $x, y, z \in A$,

$$(x \nwarrow_1 y) \nwarrow_1 z = x \nwarrow_1 (y * z), (x \nearrow_1 y) \nwarrow_1 z = x \nearrow_1 (y \ll z), (x \wedge_1 y) \nearrow_1 z = x \nearrow_1 (y \gg z), \quad (4.4.1)$$

$$(x \swarrow_1 y) \nwarrow_1 z = x \swarrow_1 (y \bigwedge z), (x \searrow_1 y) \nwarrow_1 z = x \searrow_1 (y \nwarrow_{12} z), (x \vee_1 y) \nearrow_1 z = x \searrow_1 (y \nearrow_{12} z), \quad (4.4.2)$$

$$(x \prec_1 y) \swarrow_1 z = x \swarrow_1 (y \bigvee z), (x \succ_1 y) \swarrow_{12} z = x \searrow_1 (y \swarrow_{12} z), (x \Sigma_1 y) \searrow_1 z = x \searrow_1 (y \searrow_{12} z), \quad (4.4.3)$$

$$(x \nwarrow_2 y) \nwarrow_1 z = x \nwarrow_2 (y \Sigma_1 z), (x \nearrow_2 y) \nwarrow_1 z = x \nearrow_2 (y \prec_1 z), (x \wedge_2 y) \nearrow_1 z = x \nearrow_2 (y \succ_1 z), \quad (4.4.4)$$

$$(x \swarrow_2 y) \nwarrow_1 z = x \swarrow_2 (y \wedge_1 z), (x \searrow_2 y) \nwarrow_1 z = x \searrow_2 (y \nwarrow_1 z), (x \vee_2 y) \nearrow_1 z = x \searrow_2 (y \nearrow_1 z), \quad (4.4.5)$$

$$(x \prec_2 y) \swarrow_1 z = x \swarrow_2 (y \vee_1 z), (x \succ_2 y) \swarrow_1 z = x \searrow_2 (y \swarrow_1 z), (x \Sigma_2 y) \searrow_1 z = x \searrow_2 (y \searrow_1 z), \quad (4.4.6)$$

$$(x \nwarrow_{12} y) \nwarrow_2 z = x \nwarrow_2 (y \Sigma_2 z), (x \nearrow_{12} y) \nwarrow_2 z = x \nearrow_2 (y \prec_2 z), (x \bigwedge y) \nearrow_2 z = x \nearrow_2 (y \succ_2 z), \quad (4.4.7)$$

$$(x \swarrow_{12} y) \nwarrow_2 z = x \swarrow_2 (y \wedge_2 z), (x \searrow_{12} y) \nwarrow_2 z = x \searrow_2 (y \nwarrow_2 z), (x \bigvee y) \nearrow_2 z = x \searrow_2 (y \nearrow_2 z), \quad (4.4.8)$$

$$(x \ll y) \swarrow_2 z = x \swarrow_2 (y \vee_2 z), (x \gg y) \swarrow_2 z = x \searrow_2 (y \swarrow_2 z), (x * y) \searrow_2 z = x \searrow_2 (y \searrow_2 z), \quad (4.4.9)$$

where

$$x \succ_i y = x \nearrow_i y + x \searrow_i y,\ x \prec_i y = x \nwarrow_i y + x \swarrow_i y, \quad i = 1, 2; \quad (4.4.10)$$

$$x \vee_i y = x \searrow_i y + x \swarrow_i y,\ x \wedge_i y = x \nearrow_i y + x \nwarrow_i y, \quad i = 1, 2; \quad (4.4.11)$$

$$x \bigvee y = x \vee_1 y + x \vee_2 y,\ x \bigwedge y = x \wedge_1 y + x \wedge_2 y; \quad (4.4.12)$$

$$x \gg y = x \succ_1 y + x \succ_2 y,\ x \ll y = x \prec_1 y + x \prec_2 y; \quad (4.4.13)$$

$$x \circ_{12} y = x \circ_1 y + x \circ_2 y, \quad \text{with } \circ \in \{\searrow, \nearrow, \nwarrow, \searrow\}; \quad (4.4.14)$$

$$x \Sigma_1 y = x \vee_1 y + x \wedge_1 y = x \succ_1 y + x \prec_1 y,\quad x \Sigma_2 y = x \vee_2 y + x \wedge_2 y = x \succ_2 y + x \prec_2 y; \quad (4.4.15)$$

and

$$x * y = x\Sigma_1 y + x\Sigma_2 y = \sum_{i=1}^{2}(x \searrow_i y + x \nearrow_i y + x \nwarrow_i y + x \swarrow_i y). \tag{4.4.16}$$

Proposition 4.4.2. ([Le3]) *Let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an octo-algebra.*

(1) The product given by equation (4.4.14) defines a quadri-algebra $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ which is called the associated depth quadri-algebra of $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$. And $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is called a compatible octo-algebra structure on the (depth) quadri-algebra $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$.

(2) The product given by equation (4.4.10) defines a quadri-algebra $(A, \succ_2, \succ_1, \prec_1, \prec_2)$ which is called the associated vertical quadri-algebra of $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$. And $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is called a compatible octo-algebra structure on the (vertical) quadri-algebra $(A, \succ_2, \succ_1, \prec_1, \prec_2)$.

(3) The product given by equation (4.4.11) defines a quadri-algebra $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$ which is called the associated horizontal quadri-algebra of $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$. And $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is called a compatible octo-algebra structure on the horizontal quadri-algebra $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$.

(4) The product given by equation (4.4.12) defines a dendriform algebra $(A, \bigvee, \bigwedge)$. It is the associated vertical dendriform algebra of both the quadri-algebras $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ and $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$.

(5) The product given by equation (4.4.13) defines a dendriform algebra $(A, \gg, \ll)$. It is the associated horizontal dendriform algebra of both the quadri-algebras $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ and $(A, \succ_2, \succ_1, \prec_1, \prec_2)$.

(6) The product given by equation (4.4.15) defines a dendriform algebra (A, Σ_2, Σ_1). It is the associated horizontal dendriform algebra of the quadri-algebra $(A, \vee_2, \wedge_2, \wedge_1, \vee_1)$ and the associated vertical dendriform algebra of the quadri-algebra $(A, \succ_2, \succ_1, \prec_1, \prec_2)$.

(7) The product given by equation (4.4.16) defines an associative algebra $(A, *)$ which is called the associated associative algebra of $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$. And $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is called a compatible octo-algebra structure on the associative algebra $(A, *)$.

For brevity, we pay our main attention to the study of the associated depth quadri-algebras of the octo-algebras. The corresponding study on the associated vertical and horizontal quadri-algebras are completely similar.

Proposition 4.4.3. *Let A be a vector space with eight bilinear products denoted by $\searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2$: $A \otimes A \to A$. Then $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ is an octo-algebra if and only if $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ defined by equation (4.4.14) is a quadri-algebra and $(L_{\searrow_2}, R_{\searrow_1}, L_{\nearrow_2}, R_{\nearrow_1}, L_{\nwarrow_2}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\swarrow_1}, A)$ is a bimodule.*

Proof. The conclusions can be obtained from the following correspondence by substituting $(L_{\searrow_2}, R_{\searrow_1}, L_{\nearrow_2}, R_{\nearrow_1}, L_{\nwarrow_2}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\swarrow_1}, A)$ into equations (4.1.1)-(4.1.9).

$$\begin{array}{lll}
(4.1.1\text{-}1) \Longleftrightarrow (4.4.7\text{-}1); & (4.1.1\text{-}2) \Longleftrightarrow (4.4.4\text{-}1); & (4.1.1\text{-}3) \Longleftrightarrow (4.4.1\text{-}1); \\
(4.1.2\text{-}1) \Longleftrightarrow (4.4.8\text{-}1); & (4.1.2\text{-}2) \Longleftrightarrow (4.4.5\text{-}1); & (4.1.2\text{-}3) \Longleftrightarrow (4.4.2\text{-}1); \\
(4.1.3\text{-}1) \Longleftrightarrow (4.4.9\text{-}1); & (4.1.3\text{-}2) \Longleftrightarrow (4.4.6\text{-}1); & (4.1.3\text{-}3) \Longleftrightarrow (4.4.3\text{-}1); \\
(4.1.4\text{-}1) \Longleftrightarrow (4.4.7\text{-}2); & (4.1.4\text{-}2) \Longleftrightarrow (4.4.4\text{-}2); & (4.1.4\text{-}3) \Longleftrightarrow (4.4.1\text{-}2); \\
(4.1.5\text{-}1) \Longleftrightarrow (4.4.8\text{-}2); & (4.1.5\text{-}2) \Longleftrightarrow (4.4.5\text{-}2); & (4.1.5\text{-}3) \Longleftrightarrow (4.4.2\text{-}2); \\
(4.1.6\text{-}1) \Longleftrightarrow (4.4.9\text{-}2); & (4.1.6\text{-}2) \Longleftrightarrow (4.4.6\text{-}2); & (4.1.6\text{-}3) \Longleftrightarrow (4.4.3\text{-}2); \\
(4.1.7\text{-}1) \Longleftrightarrow (4.4.7\text{-}3); & (4.1.7\text{-}2) \Longleftrightarrow (4.4.4\text{-}3); & (4.1.1\text{-}3) \Longleftrightarrow (4.4.1\text{-}3); \\
(4.1.8\text{-}1) \Longleftrightarrow (4.4.8\text{-}3); & (4.1.8\text{-}2) \Longleftrightarrow (4.4.5\text{-}3); & (4.1.8\text{-}3) \Longleftrightarrow (4.4.2\text{-}3); \\
(4.1.9\text{-}1) \Longleftrightarrow (4.4.9\text{-}3); & (4.1.9\text{-}2) \Longleftrightarrow (4.4.6\text{-}3); & (4.1.9\text{-}3) \Longleftrightarrow (4.4.3\text{-}3).
\end{array}$$

□

Corollary 4.4.4. *Let* $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ *be an octo-algebra.*

(1) $(L_{\searrow_2}, R_{\nearrow_1}, L_{\swarrow_2}, R_{\nwarrow_1}, A)$ *is a bimodule of the dendriform algebra* $(A, \gg, \ll)$.

(2) $(L_{\searrow_2}, R_{\nwarrow_1}, A)$ *is a bimodule of the associated associative algebra* $(A, *)$.

Proof. (1) follows from Propositions 4.1.2, 4.4.2 and 4.4.3. (2) follows from Corollary 4.1.3 and Proposition 4.4.3. □

Corollary 4.4.5. *Let* $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ *be an octo-algebra and* $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ *be the associated depth quadri-algebra. Then*

$$(L_{\searrow_{12}}, R_{\searrow_{12}}, L_{\nearrow_{12}}, R_{\nearrow_{12}}, L_{\nwarrow_{12}}, R_{\nwarrow_{12}}, L_{\swarrow_{12}}, R_{\swarrow_{12}}, A),\ (L_{\searrow_{12}}, 0, 0, R_{\nearrow_{12}}, 0, R_{\nwarrow_{12}}, L_{\swarrow_{12}}, 0, A),$$

$$(L_{\nearrow_{12}} + L_{\searrow_{12}}, 0, 0, R_{\nearrow_{12}} + R_{\searrow_{12}}, 0, R_{\nwarrow_{12}} + R_{\swarrow_{12}}, L_{\nwarrow_{12}} + L_{\swarrow_{12}}, 0, A),$$

$$(L_{\searrow_2}, R_{\searrow_1}, L_{\nearrow_2}, R_{\nearrow_1}, L_{\nwarrow_2}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\swarrow_1}, A),\ (L_{\searrow_2}, 0, 0, R_{\nearrow_1}, 0, R_{\nwarrow_1}, L_{\swarrow_2}, 0, A),$$

$$\text{and}\ \ (L_{\nearrow_2} + L_{\searrow_2}, 0, 0, R_{\nearrow_1} + R_{\searrow_1}, 0, R_{\nwarrow_1} + R_{\swarrow_1}, L_{\nwarrow_2} + L_{\swarrow_2}, 0, A),$$

are bimodules of $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$*. On the other hand,*

$$(R^*_*, L^*_{\nwarrow_{12}}, -R^*_{\bigvee}, -L^*_{\ll}, R^*_{\searrow_{12}}, L^*_*, -R^*_{\gg}, -L^*_{\bigwedge}, A^*),\ (R^*_*, 0, 0 - L^*_{\ll}, 0, L^*_*, -R^*_{\gg}, 0, A^*),$$

$$(R^*_{\bigwedge}, 0, 0, -L^*_{\swarrow_{12}}, 0, L^*_{\bigvee}, -R^*_{\nearrow_{12}}, 0, A^*),$$

$$(R^*_{\Sigma_1}, L^*_{\nwarrow_2}, -R^*_{\vee_1}, -L^*_{\prec_2}, R^*_{\searrow_1}, L^*_{\Sigma_2}, -R^*_{\succ_1}, -L^*_{\wedge_2}, A^*),\ (R^*_{\Sigma_1}, 0, 0 - L^*_{\prec_2}, 0, L^*_{\Sigma_2}, -R^*_{\succ_1}, 0, A^*),$$

$$\text{and}\ \ (R^*_{\wedge_1}, 0, 0, -L^*_{\swarrow_2}, 0, L^*_{\vee_2}, -R^*_{\nearrow_1}, 0, A^*)$$

are bimodules of $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$*, too.*

Proof. It follows from Propositions 4.1.2 and 4.1.4, Example 4.1.6 and Proposition 4.4.3. □

Proposition 4.4.6. *Let* $T : V \to A$ *be an* $\mathcal{O}$*-operator of a quadri-algebra* $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ *associated to a bimodule* $(l_\searrow, r_\searrow, l_\nearrow, r_\nearrow, l_\nwarrow, r_\nwarrow, l_\swarrow, r_\swarrow, V)$*. Then there exists an octo-algebra structure on* V *given by*

$$u \searrow_1 v = r_\searrow(T(v))u,\ u \searrow_2 v = l_\searrow(T(u))v,\ u \nearrow_1 v = r_\nearrow(T(v))u,\ u \nearrow_2 v = l_\nearrow(T(u))v,$$

$$u \nwarrow_1 v = r_\nwarrow(T(v))u,\ u \nwarrow_2 v = l_\nwarrow(T(u))v,\ u \swarrow_1 v = r_\swarrow(T(v))u,\ u \swarrow_2 v = l_\swarrow(T(u))v, \tag{4.4.17}$$

for any $u, v \in V$. Therefore there exists a quadri-algebra structure on V given by equation (4.4.14) and T is a homomorphism of quadri-algebras. Furthermore, $T(V) = \{T(v)|v \in V\} \subset A$ is a quadri-subalgebra of A and there is an induced octo-algebra structure on $T(V)$ given by

$$T(u) \searrow_1 T(v) = T(u \searrow_1 v),\ T(u) \searrow_2 T(v) = T(u \searrow_2 v),\ T(u) \nearrow_1 T(v) = T(u \nearrow_1 v),$$
$$T(u) \nearrow_2 T(v) = T(u \nearrow_2 v),\ T(u) \nwarrow_1 T(v) = T(u \nwarrow_1 v),\ T(u) \nwarrow_2 T(v) = T(u \nwarrow_2 v),$$
$$T(u) \swarrow_1 T(v) = T(u \swarrow_1 v),\ T(u) \swarrow_2 T(v) = T(u \swarrow_2 v),\ ,\ \forall u, v \in V. \tag{4.4.18}$$

Moreover, its corresponding associated depth quadri-algebra structure on $T(V)$ given by equation (4.4.14) is just the quadri-subalgebra structure of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ and T is a homomorphism of quadri-algebras.

Proof. The proof is similar as of Proposition 3.4.6, where a similar correspondence is given in the proof of Proposition 4.4.3. □

Definition 4.4.7. ([Le3]) Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. An $\mathcal{O}$-operator R of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to the regular bimodule $(L_\searrow, R_\searrow, L_\nearrow, R_\nearrow, L_\nwarrow, R_\nwarrow, L_\swarrow, R_\swarrow, A)$ is called a Rota-Baxter operator on $(A, \searrow, \nearrow, \nwarrow, \swarrow)$, that is, R satisfies (for any $x, y \in A$)

$$R(x \searrow y) = R(R(x) \searrow y + x \searrow R(y)),\ R(x \nearrow y) = R(R(x) \nearrow y + x \nearrow R(y)),$$
$$R(x \nwarrow y) = R(R(x) \nwarrow y + x \nwarrow R(y)),\ R(x \swarrow y) = R(R(x) \swarrow y + x \swarrow R(y)). \tag{4.4.19}$$

By Proposition 4.4.6, the following conclusion follows immediately.

Corollary 4.4.8. ([Le3], Proposition 5.12) *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra and R be a Rota-Baxter operator on $(A, \searrow, \nearrow, \nwarrow, \swarrow)$. Then there exists an octo-algebra structure on A defined by (for any $x, y \in A$)*

$$x \searrow_1 y = x \searrow R(y),\ x \searrow_2 y = R(x) \searrow y,\ x \nearrow_1 y = x \nearrow R(y),\ x \nearrow_2 y = R(x) \nearrow y,$$
$$x \nwarrow_1 y = x \nwarrow R(y),\ x \nwarrow_2 y = R(x) \nwarrow y,\ x \swarrow_1 y = x \swarrow R(y),\ x \swarrow_2 y = R(x) \swarrow y. \tag{4.4.20}$$

Lemma 4.4.9. ([AL], Proposition 2.5 and [Le3], Proposition 5.13) *Let R_1, R_2 and R_3 be three pairwise commutating Rota-Baxter operators on an associative algebra $(A, *)$. Then R_2 is a Rota-Baxter operator on the dendriform algebra obtained from R_3 by Corollary 2.1.9. Moreover, R_1 is a Rota-Baxter operator on the quadri-algebra obtained from the above dendriform algebra and R_2 by Corollary 3.4.8.*

Therefore by Corollaries 2.1.9, 3.4.8 and 4.4.8, it is obvious that the Rota-Baxter operators on associative algebras also can construct octo-algebras as follows.

Corollary 4.4.10. *Let R_1, R_2 and R_3 be three pairwise commuting Rota-Baxter operators on an associative algebra $(A, *)$. Then there exists an octo-algebra structure on A defined by*

$$x \searrow_1 y = R_2R_3(x) * R_1(y),\ x \searrow_2 y = R_1R_2R_3(x) * y,\ x \nearrow_1 y = R_2(x) * R_1R_3(y),$$
$$x \nearrow_2 y = R_1R_2(x) * R_3(y), x \swarrow_1 y = R_3(x) * R_1R_2(y),\ x \swarrow_2 y = R_1R_3(x) * R_2(y),$$

$$x \nwarrow_1 y = x * R_1R_2R_3(y),\ x \nwarrow_2 y = R_1(x) * R_2R_3(y),\ \ \forall x, y \in A. \tag{4.4.21}$$

Corollary 4.4.11. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. Then there exists a compatible octo-algebra structure on A such that $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is the associated depth quadri-algebra if and only if there exists an invertible $\mathcal{O}$-operator T of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to certain bimodule $(l_\searrow, r_\searrow, l_\nearrow, r_\nearrow, l_\nwarrow, r_\nwarrow, l_\swarrow, r_\swarrow, V)$ (hence $\dim V = \dim A$).*

Proof. The compatible octo-algebra structure on $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is given by

$$x \searrow_1 y = T(r_\searrow(y)T^{-1}(x)),\ x \searrow_2 y = T(l_\searrow(x)T^{-1}(y)),\ x \nearrow_1 y = T(r_\nearrow(y)T^{-1}(x)),$$

$$x \nearrow_2 y = T(l_\nearrow(x)T^{-1}(y)),\ x \swarrow_1 y = T(r_\swarrow(y)T^{-1}(x)),\ x \swarrow_2 y = T(l_\swarrow(x)T^{-1}(y)),$$

$$x \nwarrow_1 y = T(r_\nwarrow(y)T^{-1}(x)),\ x \nwarrow_2 y = T(l_\nwarrow(x)T^{-1}(y)),\ \ \forall x, y \in A.$$

Conversely, let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an octo-algebra and $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ be the associated depth quadri-algebra. Then $(L_{\searrow_2}, R_{\searrow_1}, L_{\nearrow_2}, R_{\nearrow_1}, L_{\nwarrow_2}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\swarrow_1}, A)$ is a bimodule of $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ and the identity map id is an $\mathcal{O}$-operator of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to it. □

Proposition 4.4.12. *Let $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ be a quadri-algebra. If there is a nondegenerate skew-symmetric 2-cocycle ω of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$, then there exists a compatible octo-algebra structure on $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ defined by (for any $x, y, z \in A$)*

$$\omega(x \searrow_1 y, z) = \omega(x, y \nwarrow z),\ \ \omega(x \searrow_2 y, z) = \omega(y, z * x),$$

$$\omega(x \nearrow_1 y, z) = \omega(x, -y \prec z),\ \ \omega(x \searrow_2 y, z) = \omega(y, -z \vee x),$$

$$\omega(x \nwarrow_1 y, z) = \omega(x, y * z),\ \ \omega(x \nwarrow_2 y, z) = \omega(y, z \searrow x),$$

$$\omega(x \swarrow_1 y, z) = \omega(x, -y \wedge z),\ \ \omega(x \swarrow_2 y, z) = \omega(y, -z \succ x), \tag{4.4.22}$$

such that $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is the associated depth quadri-algebra.

Proof. The proof is similar as of Proposition 3.4.11. Here the invertible linear map $T : A \to A^*$ is defined by $\langle T(x), y\rangle = \omega(x, y)$. Then T^{-1} is an invertible $\mathcal{O}$-operator of $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ associated to the bimodule $(R^*_*, L^*_\nwarrow, -R^*_\vee, -L^*_\prec, R^*_\searrow, L^*_*, -R^*_\succ, -L^*_\wedge, A^*)$. □

Corollary 4.4.13. (cf. Corollary 4.2.10) *Let $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ be an octo-algebra and $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ be the associated depth quadri-algebra. Then r given by equation (2.2.9) is a skew-symmetric solution of Q-equation in the quadri-algebra*

$$A \ltimes_{R^*_{\Sigma_1}, L^*_{\nwarrow_2}, -R^*_{\vee_1}, -L^*_{\prec_2}, R^*_{\searrow_1}, L^*_{\Sigma_2}, -R^*_{\succ_1}, -L^*_{\wedge_2}} A^*,$$

where $\{e_1, \cdots, e_n\}$ is a basis of A and $\{e_1^*, \cdots, e_n^*\}$ is its dual basis. Moreover there is a natural 2-cocycle of the quadri-algebra $A \ltimes_{R^*_{\Sigma_1}, L^*_{\nwarrow_2}, -R^*_{\vee_1}, -L^*_{\prec_2}, R^*_{\searrow_1}, L^*_{\Sigma_2}, -R^*_{\succ_1}, -L^*_{\wedge_2}} A^*$, induced by $r^{-1} : A \oplus A^* \to (A \oplus A^*)^*$, which is given by equation (2.2.10).

Proof. The proof is similar as of Proposition 3.4.12. Note that here

$$(R^*_{\Sigma_1}, L^*_{\nwarrow_2}, -R^*_{\vee_1}, -L^*_{\prec_2}, R^*_{\searrow_1}, L^*_{\Sigma_2}, -R^*_{\succ_1}, -L^*_{\wedge_2}, A^*)$$

is the dual bimodule of the bimodule $(L_{\searrow_2}, R_{\searrow_1}, L_{\nearrow_2}, R_{\nearrow_1}, L_{\nwarrow_2}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\swarrow_1}, A)$ of the associated depth quadri-algebra $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ and id is an $\mathcal{O}$-operator of $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ associated to the bimodule $(L_{\searrow_2}, R_{\searrow_1}, L_{\nearrow_2}, R_{\nearrow_1}, L_{\nwarrow_2}, R_{\nwarrow_1}, L_{\swarrow_2}, R_{\swarrow_1}, A)$. □

Proposition 4.4.14. *Let* $(A, \searrow_1, \searrow_2, \nearrow_1, \nearrow_2, \nwarrow_1, \nwarrow_2, \swarrow_1, \swarrow_2)$ *be an octo-algebra and* $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ *be the associated depth quadri-algebra. Let* $r \in A \otimes A$ *be symmetric. Then* r *is an* $\mathcal{O}$*-operator of* $(A, \searrow_{12}, \nearrow_{12}, \nwarrow_{12}, \swarrow_{12})$ *associated to the bimodule*

$$(R^*_{\Sigma_1}, L^*_{\nwarrow_2}, -R^*_{\vee_1}, -L^*_{\prec_2}, R^*_{\searrow_1}, L^*_{\Sigma_2}, -R^*_{\succ_1}, -L^*_{\wedge_2}, A^*)$$

if and only if r *satisfies*

$$r_{13} \searrow r_{23} = r_{23}\Sigma_1 r_{12} + r_{12} \nwarrow_2 r_{13}, \tag{4.4.23}$$

$$r_{13} \nearrow r_{23} = -r_{23} \vee_1 r_{12} - r_{12} \prec_2 r_{13}, \tag{4.4.24}$$

$$r_{13} \nwarrow r_{23} = r_{23} \searrow_1 r_{12} + r_{12}\Sigma_2 r_{13}, \tag{4.4.25}$$

$$r_{13} \swarrow r_{23} = -r_{23} \succ_1 r_{12} - r_{12} \wedge_2 r_{13}. \tag{4.4.26}$$

Proof. The proof is similar as of Proposition 3.4.13. □

Remark 4.4.15. From the study in the previous sections, it is enough reasonable to regard a set of equations (4.4.23)-(4.2.26) in an octo-algebra as an analogue of the classical Yang-Baxter equation ([NB]). We call it *O-equation* in an octo-algebra.

5. Summary and generalization

5.1. **Summary.** We can summarize the study in the previous sections as follows.

(I) Algebra structures. The relations between the different algebras appearing in this paper can be summarized by the following diagram.

$$\begin{array}{ccc}
\boxed{\text{Associative algebras}} & \overset{\mathcal{O}-\text{operator}}{\longleftrightarrow} & \boxed{\text{Dendriform algebras}} \\
& \overset{\mathcal{O}-\text{operator}}{\longleftrightarrow} & \boxed{\text{Quadri-algebras}} \\
& \overset{\mathcal{O}-\text{operator}}{\longleftrightarrow} & \boxed{\text{Octo-algebras}}
\end{array}$$

Here the meaning of $\overset{\mathcal{O}-\text{operator}}{\longleftrightarrow}$ is given as follows.

(a) $\overset{\mathcal{O}-\text{operator}}{\longrightarrow}$: An arrow-ending algebra can be obtained from an arrow-starting algebra by an $\mathcal{O}$-operator of the arrow-starting algebra (see Theorem 2.1.8, Proposition 3.4.6 and Proposition 4.4.6). As special cases, the construction of the arrow-ending algebras from the arrow-starting algebras by the Rota-Baxter operators follows immediately ([Ag2],[AL],[Le3]).

(b) $\overset{\mathcal{O}-\text{operator}}{\longleftarrow}$: The existence of a compatible arrow-starting algebra on an arrow-ending algebra is decided by the existence of an invertible $\mathcal{O}$-operator of the arrow-ending algebra (see Corollary 2.1.10, Corollary 3.4.10, Corollary 4.4.11). In particular, id is an $\mathcal{O}$-operator of the

arrow-ending algebra associated to certain bimodule given by the multiplication operators (or equivalently, the definition identities) of the arrow-starting algebras (see Corollary 2.1.7, Proposition 3.4.3 and Proposition 4.4.3).

(II) Algebraic equations (analogues of the classical Yang-Baxter equation) and bilinear forms. There is a "chain" of algebraic equations and bilinear forms on the associative algebras, dendriform algebras and quadri-algebras corresponding to the algebra relations given in the above (I). It can be interpreted explicitly by the following diagrams.

Associative algebra
Symmetric bilinear form
Invariant

$\updownarrow$ D.C.

Associative algebra
Associative Yang-Baxter equation
Skew-symmetric solution
Skew-symmetric part of an $\mathcal{O}$-operator
$\mathcal{O}$-operator associated to (R^*, L^*)

$\Longrightarrow$

Associative algebra
Skew-symmetric bilinear form
Connes cocycle

$\Longrightarrow$

Dendriform algebra

$\updownarrow$ D.C.

Associative algebra
D-equation in a dendriform algebra
Symmetric solution
Symmetric part of an $\mathcal{O}$-operator
$\mathcal{O}$-operator associated to $(R^*_{\prec}, L^*_{\succ})$

Part (AI) Part (AII) Part (AIII)

Dendriform algebra
Skew-symmetric bilinear form
Invariant

$\updownarrow$ D.C.

Dendriform algebra
D-equation in a dendriform algebra
Symmetric solution
Symmetric part of an $\mathcal{O}$-operator
$\mathcal{O}$-operator associated to $(R_*^*, -L_\prec^*, -R_\succ^*, L_*^*)$

$\Longrightarrow$

Dendriform algebra
Symmetric bilinear form
2-cocycle

$\Longrightarrow$

Quadri-algebra

$\updownarrow$

Dendriform algebra
Q-equation in a quadri-algebra
Skew-symmetric solution
Skew-symmetric part of an $\mathcal{O}$-operator
$\mathcal{O}$-operator associated to $(R_\wedge^*, -L_\swarrow^*, -R_\nearrow^*, L_\vee^*)$

Part (DI) Part (DII) Part (DIII)

Quadri-algebra
Symmetric bilinear form
Invariant

$\updownarrow$

Quadri-algebra
Q-equation in a quadri-algebra
Skew-symmetric solution
Skew-symmetric part of an $\mathcal{O}$-operator
$\mathcal{O}$-operator associated to $(R_*^*, L_\nwarrow^*, -R_\vee^*, -L_\prec^*, R_\searrow^*, L_*^*, -R_\succ^*, -L_\wedge^*)$

$\Longrightarrow$

Quadri-algebra
Skew-symmetric bilinear form
2-cocycle

$\Longrightarrow$

Octo-algebra

$\updownarrow$

Quadri-algebra
O-equation in an octo-algebra
Symmetric solution
$\mathcal{O}$-operator associated to $(R_{\Sigma_1}^*, L_{\nwarrow_2}^*, -R_{\vee_1}^*, -L_{\prec_2}^*, R_{\searrow_1}^*, L_{\Sigma_2}^*, -R_{\succ_1}^*, -L^{)}_{\wedge_2}$

Part (QI) Part (QII) Part (QIII)

Here "D.C." is the abbreviation of "double construction" and "$\Longrightarrow$" is under the invertible cases. Moreover, we have the following equivalences.

$$\text{Part (AII)} \Longleftrightarrow \text{Part (DI)}; \qquad \text{Part (DII)} \Longleftrightarrow \text{Part (QI)}.$$

A simplified illustration can be expressed as follows.

Symmetric bilinear form **Associative algebra**	$\xrightarrow{AYBE}$	Skew-symmetric bilinear form **Associative algebra** **Dendriform algebra**
	$\xrightarrow{D-\text{equation}}$	Symmetric bilinear form **Dendriform algebra** **Quadri-algebra**
	$\xrightarrow{Q-\text{equation}}$	Skew-symmetric bilinear form **Quadri-algebra** **Octo-algebra**

5.2. **Generalization.** It is of course natural to continue and extend the study in the previous sections to octo-algebras (to study their $\mathcal{O}$-operators on and the related topics) and even the algebra systems with more operations in an associative cluster. In fact, we can give an outline of such a study by induction. Thus, starting from the associative algebras, a similar study as in this paper can be easily given for any algebra with 2^k operations in an associative cluster.

Let $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$ be an algebra with 2^n operations in an associative cluster. Suppose that there has already been a similar study as follows on the algebras with 2^m operations for any $m < n$. In particular, a bilinear form on A satisfying certain conditions with fixed symmetry (see the following step (5)) is known due to our assumption that the theory starts from the associative algebras with symmetric invariant bilinear forms. Without loss of generality, we assume that it is symmetric. Then the whole study can be divided into 6 steps and the explicit procedure is given as follows.

Step (1) Give the bimodule structures of $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$. It can be done by the way of semidirect sum ([Sc]). It is easy to get the relations of between them and the bimodules of the algebras with 2^m operations for any $m < n$. Then the dual bimodule structure (see the proof of Proposition 4.1.4) can be found.

Step (2) Take the 2^{n+1} operations on A denoted by

$$\circ_1^1, \circ_1^2, \circ_2^1, \circ_2^2, \cdots, \circ_{2^n}^1, \circ_{2^n}^2 : A \otimes A \to A.$$

The definition identity of the algebra $(A, \circ_1^1, \circ_1^2, \circ_2^1, \circ_2^2, \cdots, \circ_{2^n}^1, \circ_{2^n}^2)$ is decided by

$$x \circ_i y = x \circ_i^1 y + x \circ_i^2 y, \ \ \forall x, y \in A$$

and $(L_{\circ_1^2}, R_{\circ_1^1}, \cdots, L_{\circ_{2^n}^2}, R_{\circ_{2^n}^1}, A)$ is a bimodule of $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$.

Step (3) Define the $\mathcal{O}$-operators of $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$ by a natural and obvious way. Then the relations between the two algebras, in particular, the construction of $(A, \circ_1^1, \circ_1^2, \circ_2^1, \circ_2^2, \cdots, \circ_{2^n}^1, \circ_{2^n}^2)$ from $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$ with an $\mathcal{O}$-operator of the latter algebra and the existence of

the former algebra in the latter algebra are given as we have summarized in (I) in subsection 5.1.

Step (4) The skew-symmetric $\mathcal{O}$-operator of $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$ associated to the dual bimodule of the regular bimodule $(L_{\circ_1}, R_{\circ_1}, \cdots, L_{\circ_{2^n}}, R_{\circ_{2^n}}, A)$ of $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$ gives an explicit algebraic equation (or a set of equations) in $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$, which can be regarded as an analogue of the classical Yang-Baxter equation.

Step (5) The skew-symmetric invertible solution of the equation obtained in the step (4) gives a skew-symmetric bilinear form satisfying certain conditions. Moreover, such a nondegenerate skew-symmetric bilinear form can induce an algebra $(A, \circ_1^1, \circ_1^2, \circ_2^1, \circ_2^2, \cdots, \circ_{2^n}^1, \circ_{2^n}^2)$ and the skew-symmetric bilinear form is just what we try to find for both

$$(A, \circ_1, \circ_2, \cdots, \circ_{2^n}) \text{ and } (A, \circ_1^1, \circ_1^2, \circ_2^1, \circ_2^2, \cdots, \circ_{2^n}^1, \circ_{2^n}^2).$$

Step (6) The symmetric $\mathcal{O}$-operator of $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$ associated to the dual bimodule of the bimodule

$$(L_{\circ_1^2}, R_{\circ_1^1}, \cdots, L_{\circ_{2n}^2}, R_{\circ_{2n}^1}, A)$$

can give an algebraic equation (or a set of equations) in $(A, \circ_1^1, \circ_1^2, \circ_2^1, \circ_2^2, \cdots, \circ_{2^n}^1, \circ_{2^n}^2)$ which can also be regarded as an analogue of the classical Yang-Baxter equation (it coincides with the equation given in the step (4) for the algebra $(A, \circ_1^1, \circ_1^2, \circ_2^1, \circ_2^2, \cdots, \circ_{2^n}^1, \circ_{2^n}^2)$).

It is also reasonable to give the following conjecture, which may indicate an application of the analogues of the classical Yang-Baxter equation.

Conjecture There should be a similar double construction of the nondegenerate bilinear forms corresponding to the algebraic equation given in the above step (4) for any algebra $(A, \circ_1, \circ_2, \cdots, \circ_{2^n})$ as we have shown in subsections 2.2, 2.3 and 3.2 for associative algebras and dendriform algebras ([Bai3]). It has been proved to be true for quadri-algebras ([Ni]) and octo-algebras ([NB]).

As we have pointed out in the Introduction, it is likely to give an operadic interpretation for the study in this paper and the further development on this subject.

Acknowledgements

The author thanks Professors M. Aguiar, L. Guo and the referee for important suggestion. This work was supported in part by the National Natural Science Foundation of China (10621101), NKBRPC (2006CB805905), SRFDP (200800550015).

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